ELEMENTARY GEOMETRY.

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ELEMENTARY GEOMETRY

PART II.

THE CIRCLE AND PROPORTION.

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INTRODUCTION.

Def. 1. If a point moves in a plane so that its distance from a fixed point is constant, it traces out a line which is called *the circumference of a circle.*

Def. 2. The fixed point is called *the centre* of the circle.

Def. 3. The distance of any point on the circumference from the centre is called *the radius.*

Def. 4. A line through the centre, terminated both ways by the circumference, is called *a diameter* of a circle.

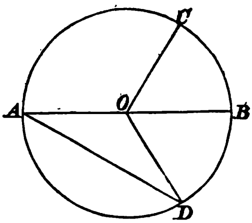
Def. 5. Any portion of a circumference is called *an arc.*

Def. 6. The figure enclosed by an arc and the radii to its extremities is called *a sector.*

Def. 7. The line joining the extremities of an arc is called *a chord* of that arc.

Def. 8. The parts into which a chord divides a circle are called *segments.*

Several properties of a circle follow at once from the definitions.



1. *All radii of a circle are equal.*

2. *All diameters of a circle are equal.*

3. *A circle cannot cut a straight line in more points than two: for from the centre of the circle not more than two equal straight lines can be drawn to meet the given straight line.*

4. *If a circle were to rotate round its centre its circumference would always occupy the same position.*

5. *Any arc is superposable on an equal arc of the same circle.*

6. *Circles are equal whose radii are equal.*

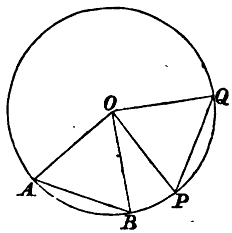
7. *A point is outside, on, or inside the circumference according as its distance from the centre is greater than, equal to, or less than the radius.*

SECTION I.

PROPERTIES OF CENTRE.

THEOREM I. ROTATORY PROPERTIES OF THE CIRCLE.

Equal arcs of a circle subtend equal angles at the centre, and have equal chords; and conversely, equal angles at the centre cut off equal arcs and have equal chords; and equal chords in a circle cut off equal arcs, and subtend equal angles at the centre.



Let $ABPQ$ be a circle, and let the *arc* AB be given equal to arc PQ, then will the angle AOB at the centre = the angle POQ, and the chord AB = the chord PQ.

For if the sector AOB were to rotate round O till A fell on P, B would fall on Q, (since arc AB= arc PQ), and the angle AOB would coincide with POQ, and therefore is equal to it; and for the same reason the chord AB=the chord PQ.

In the same manner it may be shewn that if the *angle* AOB is given equal to POQ, then the arcs and chords would coincide and are therefore equal.

And if the *chord* AB is given equal to the chord PQ, then the triangles AOB, POQ have the three sides of the one equal to the three sides of the other; and therefore the

angle $AOB =$ the angle POQ; and therefore also the arc $AB =$ the arc PQ.

COR. 1. *If the arcs or angles of sectors of a circle are equal, the sectors are equal.*

COR. 2. *In equal circles equal arcs subtend equal angles at the centre and cut off equal chords.*

COR. 3. *The diameter divides the circle into two equal parts, which are therefore called semicircles.*

REMARK ON THEOREM 1.

The student has now had several examples of theorems, and their *converse* and *opposite* theorems, and it will be well for him to observe *under what conditions a converse theorem is true.*

This may be generalized into the following statement. If A, B, C... as conditions involve D as a result, and the failure of C involves a failure of D; then A, B, D... as conditions involve C as a result.

For example, in a circle, (A), angles at the centre, (B) which stand on equal arcs, (C), are equal, (D),

and if the arcs are unequal, {the failure of C}, the angles are unequal, {the failure of D}.

Hence it follows logically that in a circle, (A), angles at the centre, (B), which are equal (D), stand on equal arcs, (C).

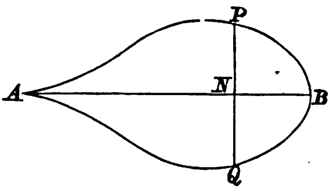
All converse theorems can be at once proved by the reductio ad absurdum, but we shall in general enunciate immediate converses, when they are true, without giving a detailed proof.

Symmetry of a figure with respect to a line.

Def. 9. A figure is said to be *symmetrical with respect to a line*, when every line at right angles to that line cuts the figure at points which are equidistant from that line.

Thus the figure $APBQ$ is symmetrical with respect to AB, if every line PNQ at right angles to AB cuts it so that

$$PN = QN.$$

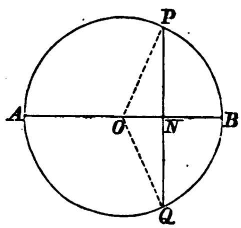


THEOREM 2.

SYMMETRY OF THE CIRCLE WITH RESPECT TO ITS DIAMETER.

A circle is symmetrical with respect to any diameter.

Let AB be any diameter, PNQ any line drawn perpendicular to AB, meeting the circle in P and Q. Then shall $PN = NQ$.

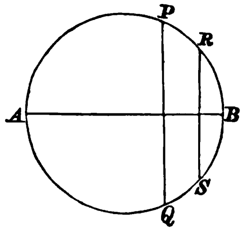


For OP and OQ are equal obliques drawn from O to PQ, and therefore they are equally distant from the perpendicular: and therefore $PN = NQ$. (Part 1, Th. 13.)

COR. 1. *Hence the semicircle APB if folded over the line AB, would coincide with the semicircle AQB, PN falling on NQ; and therefore the arc AP = the arc AQ, and the arc PB = the arc BQ.*

COR. 2. *Hence also parallel chords in a circle intercept equal arcs.*

If PQ, RS are parallel chords, then will the arc $PR =$ the arc QS.



For when one semicircle is folded on the other, P would coincide with Q, and R with S, and therefore the arc PR with the arc QS.

COR. 3. This fundamental property of a circle, viz. its symmetry with respect to a diameter, gives rise to many theorems. For it appears that the diameter AB in the figure of Cor. 2 fulfils six conditions: (1) It is perpendicular to the chord PQ; (2) it passes through the centre; (3) it bisects the chord; (4) it bisects the arc QAP; (5) it bisects the arc PBQ; (6) it bisects any chord parallel to PQ. *And since only one line can be drawn to fulfil any two of the above conditions, it follows that a line which fulfils any two of them, fulfils the remaining four.*

For example: the original theorem, with its first corollary, is (α) *If a line fulfils* (1) *and* (2), *it also fulfils* (3), (4) *and* (5).

Hence (β) *If a line bisects a chord at right angles* (1) *and* (3), *it must pass through the centre* (2).

And, (γ) *A line that bisects any chord and its arc,* (3) *and* (4), *will pass through the centre* (2).

And, (δ) *The line drawn from the centre to bisect a chord* (2 *and* 3) *is perpendicular to that chord* (1).

By combining these data in different ways, many different theorems may be made.

EXERCISES.

1. If a straight line cut two concentric circles, the parts of it intercepted between the two circumferences will be equal.

2. Of two angles at the centre, the greater angle is subtended by the greater arc; and also by the greater chord, if the sum of the two angles is less than four right angles. Prove this and its converse propositions.

3. Perpendiculars are let fall from the extremities of a diameter on any chord, or any chord produced; shew that the feet of the perpendiculars are equally distant from the centre.

4. The locus of the points of bisection of parallel chords of a circle is the diameter at right angles to those chords.

5. If a diameter of a circle bisects a chord which does not pass through the centre, it will bisect all chords which are parallel to it.

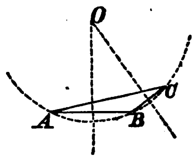
6. AB and CD are unequal parallel chords in a circle, prove that AC and BD, and likewise AD and BC intersect on the diameter perpendicular to AB and CD, or that diameter produced, and are equally inclined to that diameter.

What will be the case if AB and CD are equal?

THEOREM 3.

One circle, and only one circle, can be drawn to pass through three given points which are not in the same straight line.

Let A, B, C be the three given points. Join AB, BC.



Then since AB is to be a chord, the locus of the centre is the straight line that bisects AB at right angles. (Th. 2, Cor. β.)

Similarly, the line that bisects BC at right angles must pass through the centre. Hence the centre must be at O, the point of intersection of these perpendiculars; and the circle described with centre O and radius OA will pass through A, B and C.

And there can be only one centre, since the perpendiculars intersect in only one point.

The three points thus *determine* the circle.

COR. 1. *Circles that have three points in common, coincide wholly.*

Hence a circle is named by the letters which mark three points on its circumference.

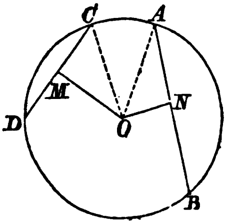
COR. 2. *Different circles can intersect in two points only.*

Def. 10. The circle is said to be *circumscribed* about the triangle ABC, and the triangle ABC is said to be *inscribed* in the circle, when the points A, B, C are on the circumference of the circle.

Def. 11. *The distance* of a chord from the centre is the perpendicular on the chord from the centre.

THEOREM 4.

Equal chords of a circle are equally distant from the centre, and conversely; and of two unequal chords the greater is nearer to the centre than the less, and conversely.



Let AB, CD be chords of a circle, OM, ON the perpendiculars on them from the centre, bisecting the chords in M and N respectively. Join OC, OA.

Then since M and N are right angles, therefore

$$OM^2 + MC^2 = OC^2,$$

and $$ON^2 + NA^2 = OA^2,$$

but $$OC = OA, \text{ and } OC^2 = OA^2;$$

therefore $$OM^2 + MC^2 = ON^2 + NA^2.$$

Hence, (1) if $AB = CD$ and $AN = CM$,

it follows that $$OM = ON.$$

(2) If $AB > CD$, $AN > CM$,

and therefore OM is $< ON$.

(3) If $ON = OM$,

therefore $AN = CM$ and $AB = CD$.

(4) If $ON < OM$,

$AN > CM$ and $AB > CD$.

COR. 1. *The diameter is the greatest chord of a circle.*

COR. 2. *The locus of the middle points of equal chords in a circle is a concentric circle.*

Exercises.

1. Given a triangle ABC to find the centre of the circumscribing circle.

2. A chord 8 inches long is drawn in a circle whose radius is 5 inches; find the distance of the chord from the centre.

3. A chord is drawn at the distance of one foot from the centre of a circle whose diameter is 26 inches; find the length of the chord.

4. Given a circle to find its centre.

5. If two equal chords intersect one another, the segments of the one are equal to the segments of the other respectively.

6. Two chords cannot bisect one another unless both pass through the centre.

7. Given a curve, to ascertain whether it is an arc of a circle or not.

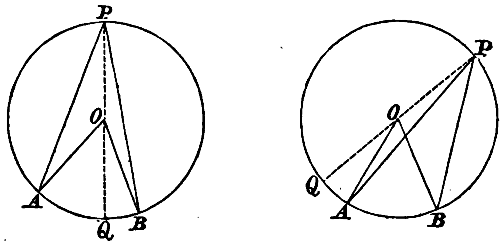
SECTION II.

Angles in Segments of the Circle.

Theorem 5.

The angle subtended at any point in the circumference by any arc of a circle is half of the angle subtended by the same arc at the centre.

Let AB be any arc, O the centre, P any point on the



circumference. Then will the angle AOB be double of the angle APB.

Join PO, and produce it to Q.

Then because, from the definition of a circle, OPA is an isosceles triangle, the angle $OAP =$ the angle OPA: but the exterior angle AOQ is equal to the two interior and opposite angles OAP and OPA, and therefore the angle AOQ is double of the angle OPA. Similarly the angle QOB is double of the angle OPB.

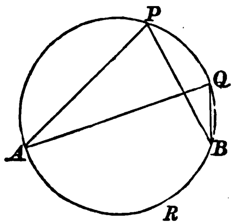
Hence (in fig. 1) the sum, or (in fig. 2) the difference of the angles AOQ, QOB is double of the sum or dif-

ference of OPA and OPB, that is, the angle AOB is double of the angle APB, or the angle APB is half of the angle AOB.

Def. 12. The angle APB is said to be the angle *in the segment* APB.

COR. 1. *Angles in the same segment of a circle are equal to one another.*

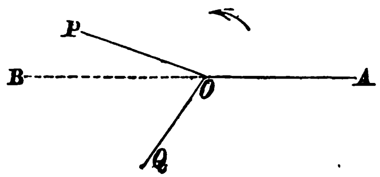
For each of the angles APB, AQB is half of the angle subtended at the centre by the arc ARB.



The segment $APQB$ is said to be *capable* of an angle equal to the angle APB or AQB.

Remark. It is important to remember here that angles may be greater than two right angles, and that the existence of such angles is contemplated in the foregoing theorem and corollary.

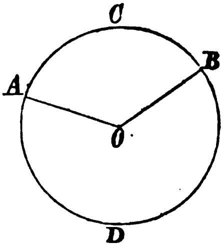
Thus, if a line be conceived to revolve round O starting from an initial position OA, and revolving in the



direction of the arrow, it describes an angle of constantly increasing magnitude; when it reaches the position OB it has described an angle equal to two right angles; and in the position OQ it has described an angle greater than

two right angles. There are thus two angles AOQ, one taken in the direction of the arrow greater than two right angles, and the other taken in the opposite direction less than two right angles.

Thus, in the figure, the angle BOA standing on the arc ACB is less than two right angles; and the angle AOB on the arc ADB is greater than two right angles.

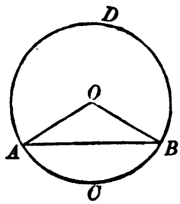


Hence it is clear that the angle at the centre standing on any arc is less than, equal to, or greater than two right angles according as the arc is less than, equal to, or greater than a semi-circumference.

COR. 2. *If a circle is divided into any two segments by a chord, the angles in the segments will be supplementary to one another.*

For of the two angles at O, the one is double of the angle in the segment ADB, and the other of the angle in the segment ACB.

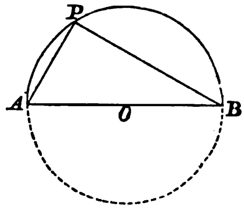
But the angles at O make up four right angles, therefore the angles in the segments ACB, ADB make up two right angles.



These segments are called *Supplementary Segments.*

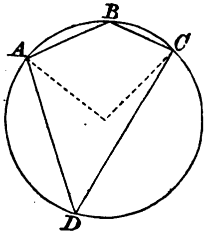
COR. 3. *The angle in a semicircle is a right angle.*

For let APB be a semicircle: then the angle in the segment is half the angle at the centre: that is, the angle APB is half the angle AOB, which is two right angles; therefore the angle APB is a right angle.



Def. 13. A polygon is said to be *inscribed in* a circle, when its angular points are on the circumference of the circle.

COR. 4. *The opposite angles of every quadrilateral figure inscribed in a circle are together equal to two right angles.*

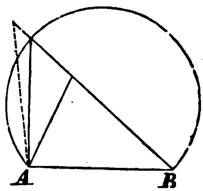


For the segments ABC, ADC are supplementary segments, and so also are the segments BAD, BCD.

Therefore by Cor. 3 the angles ABC, ADC are together equal to two right angles; and the angles BAD, BCD are also together equal to two right angles.

COR. 5. *The locus of a point at which a given straight line subtends a constant angle is an arc of a circle.*

For if AB be the given straight line, and on it there is described a segment capable of the given angle, at every point in the arc the straight line subtends the given angle, and at every other point the angle is greater or less, according as it is within or without the segment.



The segment may be described on both sides of the line AB.

It will be seen that this is the converse and opposite of Cor. 1.

Cor. 6. *In equal circles equal angles at the circumferences stand upon equal arcs, and conversely.*

This may be proved by superposition.

Exercises.

1. Prove that the lines which join the extremities of equal arcs in a circle are either equal or parallel.

2. If two opposite angles of a quadrilateral figure are together equal to two right angles, prove that a circle which passes through three of its angular points will also pass through the fourth.

3. If two opposite sides of a quadrilateral inscribed in a circle are equal, prove that the other two are parallel.

4. AB, CD are chords of a circle which cut at a constant angle. Prove that the sum of the arcs AC, BD remains constant, whatever may be the position of the chords.

5. If the diameter of a circle be one of the equal sides of an isosceles triangle, prove that its circumference will bisect the base of the triangle.

6. Circles are described on two sides of a triangle as diameters. Prove that they will intersect on the third side or third side produced.

7. Any number of chords of a circle are drawn through a point on its circumference: find the locus of their middle points.

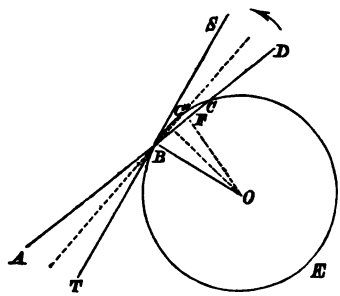
8. If through any point, within or without a circle, lines are drawn to cut the circle, prove that the locus of the middle points of the chords so formed is a circle.

SECTION III.

The Tangent and Normal.

Def. 14. When a straight line cuts a circle it is called a *secant*.

Thus *ABCD* is a secant of the circle *ACE*.



Def. 15. When one of the points in which a secant cuts a circle is made to move up to, and ultimately coincide with the other, the ultimate position of the secant is called *the tangent* at that point.

Thus, if *ABCD* be conceived to revolve round *B*, in the direction of the arrow, the point *C* will move to *C′* and will ultimately coincide with *B*, and the line *TBS*, which is the position the secant then attains, is said to be a tangent to the circle at the point *B*.

The point *B* is then called *the point of contact.*

Several important properties follow at once from this definition of a tangent.

THEOREM 6.

A tangent meets the circle in one point only, viz. the point of contact.

For since a secant can cut a circle in two points only, it follows that the parts AB, CD are wholly without the circle; and therefore when C moves up to B, and the chord BC is merged in the point B, the whole line, with exception of the point B, is outside the circle.

THEOREM 7.

The radius to the point of contact is at right angles to the tangent.

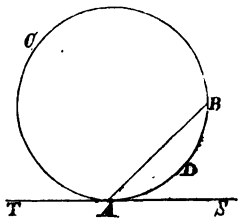
For if F be the middle point of the chord BC, OF is perpendicular to BC; and as C moves to B, F will also move up to B, and when the secant becomes a tangent, OF, which is always at right angles to the secant, coincides with the radius OB.

Therefore OB is at right angles to the tangent TBS.

COR. 1. *Hence there can be only one tangent to a circle at a given point.*

COR. 2. *The line at right angles to the tangent through the point of contact passes through the centre.*

Def. 16. When a secant is drawn from the point of contact of a tangent it divides the circle into segments which are said to be *alternate* to the angles made by the tangent with the secant on its sides opposite to the segments.

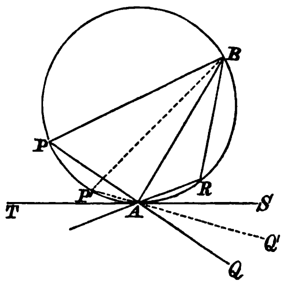


Thus ACB is a segment alternate with BAS, and BDA with BAT.

Theorem 8.

If from the point of contact of a straight line and a circle a chord of the circle be drawn, the angles made by the chord with the tangent will be equal to the angles in the alternate segments of the circle.

Let a chord AB be drawn from the point of contact A of the tangent TAS.



Then will the angles BAS, BAT be equal to the angles in the alternate segments of the circle.

For take any point P in the arc of the segment alternate to BAS, and join PB, PA, and produce PA to Q.

Conceive the point P to move along the arc towards A. Then the angle BPA in the segment remains always the same; and it will after a while assume the position of the dotted lines $BP'A$, and ultimately when P has moved up to A, the angle BPA will coincide with the angle BAS, since the limiting position of the secant PQ is the tangent AS, and BP then coincides with BA. Therefore the angle BAS = the angle BPA in the alternate segment.

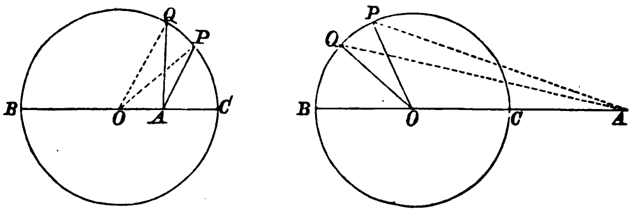
Similarly it may be shewn that the angle BAT = the angle BRA.

Def. 17. If from any point on a curve a line is drawn at right angles to the tangent at that point, it is called a *normal to the curve at that point.*

Since the radius is at right angles to the tangent to a circle, it follows that all radii are normals to a circle.

THEOREM 9.

From any point within or without a circle except the centre, two and only two normals can be drawn, one of which is the shortest, and the other the longest line that can be drawn from that point to the circumference: and as a point moves along the circumference from the extremity of the shortest to the extremity of the longest normal, its distance from the fixed point continually increases.



Let A be the fixed point, O the centre, and let AO produced through the centre meet the circumference in B, and produced if necessary in the other direction meet it in C.

Then AB and AC are normals and are the only normals, and are respectively the longest and shortest lines that can be drawn from A to the circumference: and if P, Q are any other points such that the arc CP is less than CQ, AP shall be less than AQ.

Join OP, OQ.

Then it is clear that AB and AC are at right angles to the tangents at B and C, since A is a point in the radius OB or OC.

And if P is any other point, OP is the normal at that

point, and therefore AP is not the normal: hence two and only two normals can be drawn.

Again, in the triangle APO, the difference of OP and OA is AC, since $OP = OC$; and therefore AC is less than AP: and the sum of OP and OA is AB, since $OP = OB$, and therefore AB is greater than AP. Hence of all lines drawn from A to the circumference AC is the least and AB the greatest.

Lastly, in the triangles AOP, AOQ since the two sides AO, OP are equal to AO, OQ, but the angle AOQ greater than the angle AOP, therefore AQ is greater than AP. (Part 1, Th. 20.)

Therefore as a point moves from C to A along the arc, its distance from A continually increases.

COR. 1. *Two and only two equal straight lines can be drawn from A to the circumference, one on each side of the shortest normal.*

COR. 2. *A point from which more than two equal straight lines can be drawn to a circumference must be the centre.*

THEOREM 10.

INTERSECTION OF CIRCLES.

The line that joins the centres of two intersecting circles, or that line produced, bisects at right angles their common chord.

Fig. 1.

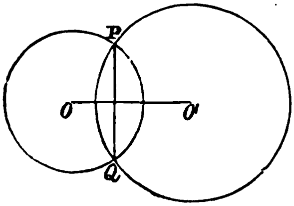
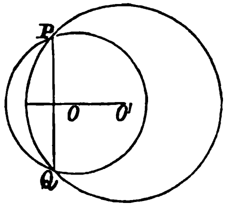


Fig. 2.



Let O, O' be the centres of two intersecting circles, PQ their common chord, then shall OO' or OO' produced bisect PQ at right angles.

For since PQ is a chord of both circles, the line which bisects PQ at right angles passes through both centres (Th. 2, β); that is, it must be the line OO'.

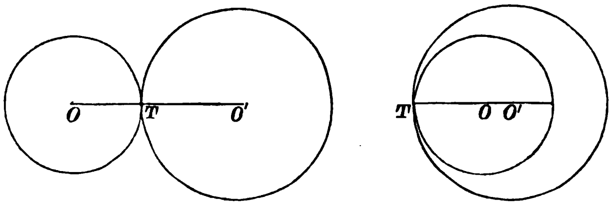
Contact of Circles.

Def. 18. When one of the points in which one circle cuts another moves up to and ultimately coincides with the other, *the circles are said to touch one another* at that point.

Since two circles intersect in only two points, it follows that two circles which touch one another can have no other point in common; for the two points of intersection are merged in the point of contact.

Theorem 11.

If two circles touch one another, the line that joins their centres will pass through the point of contact.



For if in the figures of Th. 10, the centres O, O' of the circles were to recede from one another, or were to approach one another, the points P and Q would after a while approach one another, and the chord PQ would become indefinitely small, and be merged in the point T, and the circles would touch one another at T by the definition.

But the line OO' always bisects PQ, and therefore it will ultimately pass through T the point of contact.

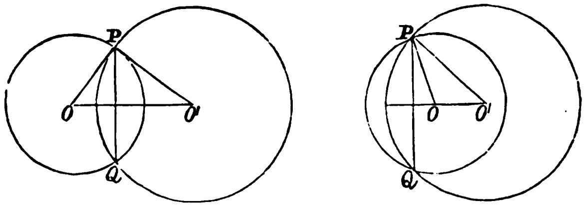
COR. 1. *Two circles that touch one another have a common tangent at their point of contact.*

For the line at right angles to OO' through T is a tangent to both circles by Th. 7.

COR. 2. *If R, r are the radii of two circles, D the distance between their centres, it follows that*

(1) *When the circles intersect,*

$$R+r>D \text{ or } R-r<D. \quad \text{Book 1, Th. 14.}$$



(2) *When the circles touch,*

$$R+r=D \text{ or } R-r=D.$$

(3) *When the circles do not meet,*

$$R+r<D \text{ or } R-r>D.$$

In other words, if R, r, D are such that any two of them are greater than the third, the circles will intersect; if two of them are together equal to the third, the circles will touch; and if two of them are together less than the third, the circles will not meet, but be wholly inside or wholly outside one another.

EXERCISES.

1. If a straight line touch the inner of two concentric circles, and be terminated by the outer, prove that it will be bisected at the point of contact.

2. Any two chords which intersect on a diameter and make equal angles with it are equal.

3. Two circles touch each other externally, and a third circle is described touching both externally. Shew that the difference of the distances of its centre from the centre of the two given circles will be constant.

4. If two circles intersect one another, and circles are drawn to touch both, prove that either the sum or the difference of the distances of their centres from the centres of the fixed circles will be constant, according as they touch (1) one internally and one externally, (2) both internally or both externally.

5. If two circles touch one another, any line through the point of contact will cut off segments from the two circles capable of the same angle.

6. If two circles touch one another, two straight lines through the point of contact will cut off arcs, the chords of which are parallel.

7. Two circles cut one another, and lines are drawn through the points of section and terminated by the circumference, shew that they intercept arcs the chords of which are parallel.

8. Circles whose radii are 6·7 and 7·8 inches are successively placed so as to have their centres 14, 14½, and 15 inches apart. Shew whether the circles will meet or touch or not meet one another.

9. What will be the case if the centres are 1 inch, 1·1 inch, or 1·2 inches apart?

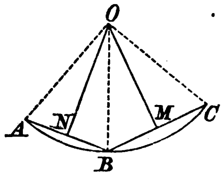
SECTION IV.

PROBLEMS.

PROBLEM 1.

Given an arc of a circle, to find the centre of the circle of which it is an arc.

Let ABC be the arc.



Construction. Draw any two chords AB, BC, and bisect them at right angles by straight lines ON, OM, intersecting at O. O shall be the centre required.

Proof. For NO is the locus of points equidistant from A and B, and therefore $AO = BO$.

Similarly, MO is the locus of points equidistant from B and C; therefore O is equidistant from A, B and C.

Hence, the circle described with centre O and radius equal to one of these three lines, will pass through the other two, and having three points coinciding with the given circular arc, must coincide with it throughout.

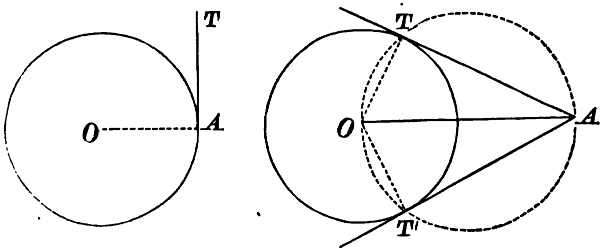
PROBLEM 2.

To draw a tangent to a circle from a given point.

There will be two cases.

First, let the given point A be on the circumference. Let O be the centre.

Construction. Join OA, and draw AT at right angles to OA.



Proof. Then AT is a tangent by Th. 7.

Secondly, let A be outside the circle.

Construction. On OA as diameter describe a circle, cutting the given circle in T and T'. Join AT, AT'; these shall be tangents from A.

Proof. For join OT, OT'. Then since ATO is a semicircle, the angle ATO is a right angle. (Th. 5, Cor. 2.) That is, AT or AT' is at right angles to the radius to the point where it meets the circumference, and therefore AT and AT' are tangents.

It may easily be proved that $AT = AT'$.

Problem 3.

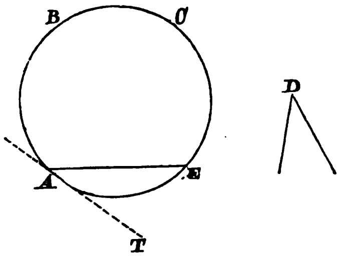
To cut from any circle a segment which shall be capable of a given angle.

Let ABC be the circle, D the given angle.

Construction. Take any point A on the circumference.

Draw AT the tangent at A; and make an angle TAE at A equal to the angle D.

Then shall AE be the chord of the segment required.

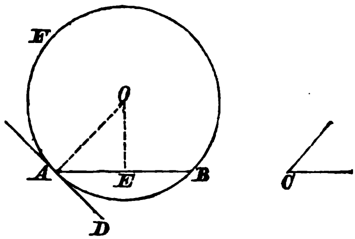


Proof. For the angle in the segment alternate to TAE is equal to the angle TAE, that is, is equal to D.

PROBLEM 4.

On a given straight line to describe a segment of a circle containing an angle equal to a given angle.

Let AB be the given line, C the given angle.



Construction. At the point A make an angle BAD equal to the angle C.

Then if a circle be described to touch AD in A, and to pass through B, the segment of that circle alternate to BAD will be the segment required.

To find the centre of this circle, draw AO at right angles to AD: then AO is the locus of the centres of all circles which touch AD at A.

And bisect AB at right angles by the line EO; then EO is the locus of the centres of circles which pass through A and B.

Therefore O, the point of intersection of these lines, is the centre of the circle required.

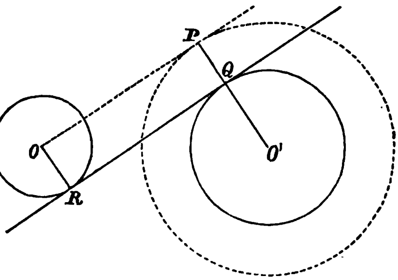
With centre O and radius OA or OB describe a circle, which will touch AD at A and pass through B, and therefore the segment AFB contains an angle equal to the angle BAD, that is to the given angle C.

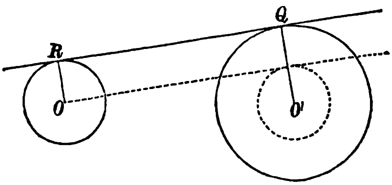
PROBLEM 5.

To draw a common tangent to two given circles.

Let the centres of the circles be O, O'.

Construction. With centre O' and radius equal to the sum or difference of the radii of the given circles, describe a circle, as in the figures.





From O draw a tangent to this circle, touching it in P. Join $O'P$, and let it, produced through P if necessary, meet the circumference of the circle whose centre is O' in the point Q. Through O draw OR parallel to PQ, and join QR. QR will be a tangent to both circles.

Proof. Since PQ is by the construction equal and parallel to OR, therefore RQ is parallel to OP. But OP is at right angles to $O'P$, since it touches the circle in P, and therefore RQ is at right angles to OR and OQ; and therefore touches both circles.

COR. 1. *When the circles are wholly outside one another, they have four common tangents: when they touch externally, they have three common tangents: when they intersect one another, they have two common tangents: when they touch internally, they have one common tangent: and when one of the circles is wholly inside the other, they have no common tangent.*

COR. 2. *By symmetry with respect to OO', pairs of the common tangents will intersect on that line.*

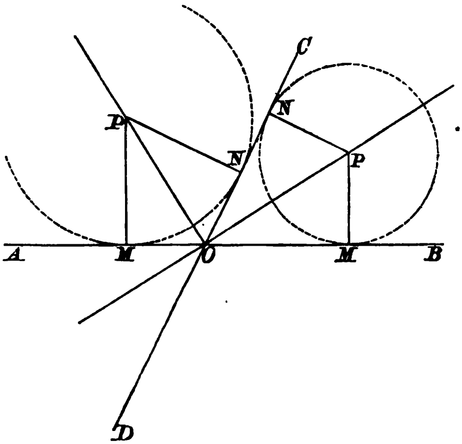
PROBLEM 6.

To find the locus of the centres of circles which touch two given straight lines.

Let AOB, COD be the two given straight lines, intersecting one another in O.

Let P be the centre of a circle which touches both the lines, PN, PM the perpendiculars from P on DOC and AOB.

Then $PN = PM$, and if OP be joined, since the triangles ONP, OMP are right angled at N and M, have the



hypothenuse OP common, and have one side $PN =$ one side PM, being radii of the same circle; therefore the triangles are equal in all respects, and the angle PON = the angle POM, that is OP bisects the angle COB.

Therefore the centres lie on the bisectors of the angles between the given lines; and these bisectors are therefore the locus required.

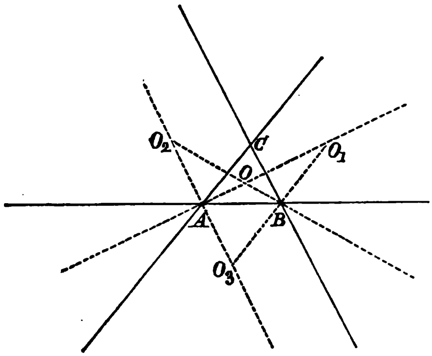
COR. 1. *It is obvious that these bisectors form two straight lines at right angles to one another.*

COR. 2. *If the given lines are parallel, the locus is a line parallel to both and equidistant from them.*

PROBLEM 7.

To describe a circle to touch three given straight lines of indefinite length.

Let the three given lines intersect in A, B, and C.



Then since the circle required is to touch the lines that intersect in A, its centre must lie on one of the bisectors of the angles at A. Similarly, it must lie on one of the bisectors of the angles at B. Therefore the construction is suggested.

Construction. Draw the bisectors of the angles at A and B, which will intersect in four points O, O_1, O_2, O_3.

These will be the centres of the circles required, and a circle described with any one of these points as centre, to touch one of the given lines, will touch the other two.

COR. 1. *It follows that COO_3 and O_2CO_1 are straight lines, that is, the six bisectors of the interior and exterior angles of a triangle intersect one another three and three in four points.*

COR. 2. *If two of the lines are parallel, only two circles can be described to touch the three lines.*

COR. 3. *If all the lines are parallel, no circle can be described to touch them all.*

SECTION V.

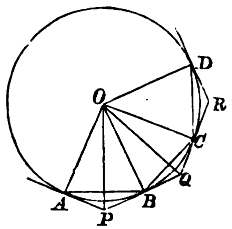
Regular Polygons.

One of the most interesting species of problem connected with the circle consists in describing regular polygons, and inscribing them in, or circumscribing them about given circles.

We shall first establish the following theorems.

Theorem 12.

If from the centre of a circle radii are drawn to make equal angles with one another consecutively all round, then if their extremities are joined consecutively, a regular polygon will be inscribed in the circle, and if at their extremities, tangents are drawn, a regular polygon will be circumscribed to the circle.



Let OA, OB, OC, OD ... be radii making equal angles consecutively to one another.

(1) Join AB, BC, CD ...

Then since the angles at O are equal, the chords which subtend them are equal, and therefore the inscribed polygon is equilateral.

And since the arcs AB, BC, CD ... are equal; therefore the arc AC = arc BD, and therefore the angle ABC in the segment ABC = the angle BCD in the segment BCD; that is, the polygon is equiangular.

Hence $ABCD$ ... will be a regular polygon inscribed in the circle.

(2) Draw tangents at A, B, C, D ... meeting one another in P, Q, R ...

Join PO, QO.

Since $AP = PB$, the line PO bisects the angle AOB, and similarly QO bisects the angle BOC. Therefore the angle $POB =$ the angle QOB, and hence $PB = BQ$ from the triangles POB, QOB.

Therefore QP is double of QB; and similarly QR is double of QC.

But $QB = QC$; and therefore $QP = QR$; and thus it may be shewn that the polygon is equilateral.

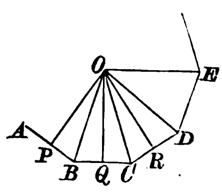
And since the angles APB, BQC ... are supplementary to the angles AOB, BOC ... they are equal to one another.

That is, the polygon is equiangular; and since PQ, QR ... are tangents, PQR ... will be a regular polygon circumscribed to the circle.

THEOREM 13.

In a regular polygon the bisectors of the angles intersect in one point, which is the centre of the circles inscribed in the polygon, or circumscribed about it.

Let $ABCDE$ ... be a regular polygon.



Bisect the angles B, C by straight lines meeting in O.

Join OD, OE ...

Since BO, CO are bisectors of the equal angles ABC, BCD, therefore the angle $OBC =$ the angle OCB; and therefore $OB = OC$.

And because in the triangles OCB, OCD, the angle

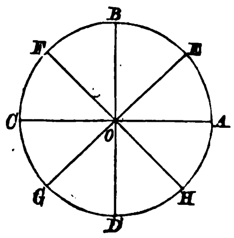
$OCB = OCD$, and the sides which contain these equal angles are equal to one another, each to each, therefore $OD = OB$, and $ODC = OBC$. But OBC is half of ABC, that is of CDE, since the polygon is equiangular. Therefore ODC is half of CDE, and therefore OD bisects the angle CDE.

In the same manner it may be shewn that $OE = OC$, and bisects the angle at E. Hence O is the centre of the circle circumscribed about the polygon, of which OB is the radius.

And because AB, BC, CD ... are equal chords in this circle, of which O is the centre, therefore the perpendiculars OP, OQ, OR on those chords are all equal; and a circle described with centre O and radius OP will pass through Q, R ... and touch AB, BC, CD ... in the points P, Q, R ... That is O is the centre of the inscribed circle of which OP is the radius.

PROBLEM 8.

To construct a regular polygon of four, eight, sixteen ... sides, and inscribe them in, or describe them about a given circle.



Take O the centre of the given circle, and draw through it two diameters at right angles to one another, meeting the circumference in A, B, C, D; then by Theorem 12, if AB, BC ... be joined, we get a regular inscribed polygon of four sides; and if tangents be drawn at A, B, C and D, we get a regular circumscribed polygon of four sides, that is a square.

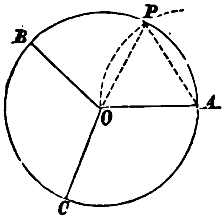
To describe an octagon, bisect (by Part I. Prob. 1) the angles AOB, BOC, COD, DOA by lines meeting the circle in E, F, G, H; then A, E, B, F... are the angular points of a regular inscribed octagon, or the points of contact of the sides of a regular circumscribed octagon.

Similarly, by again bisecting the angles at O, a regular sixteen-sided polygon may be constructed; and hence a thirty-two-sided figure, and so generally a polygon of 2^n sides, may be constructed, when n is any integer greater than 1.

PROBLEM 9.

To construct regular polygons of three, six, twelve ... sides, and inscribe them in, or describe them about a given circle.

Let O be the centre of the circle. On OA, one of the



radii, make an equilateral triangle AOP, and take AOB, double of the angle AOP.

Then since each angle of an equilateral triangle is one-third of two right angles; therefore its double is one-third of four right angles; and therefore two other equal angles BOC, COA will fill up the space round O.

Hence the angles at O being equal, A, B, C are the angular points of a regular inscribed polygon of three sides.

As before, by bisecting the angles at O we obtain the angular points of the regular hexagon; and by bisecting these angles we obtain the dodecagon; and so generally a regular polygon of 3×2^n sides may be constructed, where n may have any integral value, including zero.

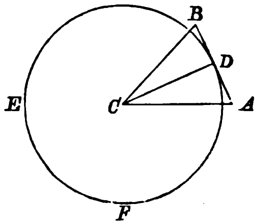
The Theorems at present proved do not enable the student to construct any regular polygons except those included in the foregoing problems.

Hereafter we shall shew that a regular pentagon can be described, and by means of it the decagon, quindecagon, &c.

THEOREM.

The area of a circle is equal to half the rectangle contained by the radius and a straight line equal to the circumference.

Let C be the centre of a circle DEF. And let AB be the side of a polygon described about the circle, CD a radius drawn to the point of contact.



Then the triangle $ABC = \frac{1}{2}$ the rectangle contained by CD and AB.

And since the whole polygon can be divided into triangles by lines drawn from the angular points to the centre,

The area of the polygon $= \frac{1}{2}$ the rectangle contained by the radius of the circle and perimeter of the polygon.

Now this is true whatever may be the number of sides of the polygon.

But by perpetually increasing the number of sides of the polygon, that polygon approaches nearer and nearer to the circle.

So finally the area of the circle $= \frac{1}{2}$ rectangle contained by radius and circumference.

Remark. This will give the area of the circle when we know the lengths of the radius and the circumference. The length of the latter cannot be found by the use of the rule and compasses. But if we added to our mathematical instruments a cylinder that might be rolled on the paper, or a tape that might be unrolled from a cylinder, we should have that length at once, and consequently the area.

MISCELLANEOUS THEOREMS AND PROBLEMS.

1. Prove that the two tangents drawn to a circle from any external point are equal.

2. If from a point without a circle two tangents *AB*, *AC* are drawn, the chord of contact *BC* will be bisected at right angles by the line from *A* to the centre.

3. If a circle is inscribed in a right-angled triangle, the excess of the two sides over the hypothenuse is equal to the diameter of the circle.

4. If a quadrilateral figure be described about a circle, the sums of the opposite sides will be equal to one another.

5. If a six-sided figure be circumscribed about a circle, the sums of the alternate sides will be equal.

6. If a quadrilateral figure be described about a circle, the angles subtended at the centre by any two opposite sides are together equal to two right angles.

7. *AOB*, *COD* are two chords of a circle at right angles to one another; prove that the squares of *OA*, *OB*, *OC*, and *OD*, are together equal to the square of the diameter.

8. The chord AB is produced both ways equally to C, D, and tangents CE, DF, drawn on opposite sides of CD; shew that EF bisects AB.

9. Two circles touch one another in A, and have a common tangent BC. Shew that the angle BAC is a right angle.

10. Describe (when possible) a circle of given radius to touch (1) two given lines, (2) two given circles.

11. Describe a circle to touch a given line in a given point, and pass through another given point.

12. Describe a circle to touch a given circle in a given point, and to pass through another given point.

13. Find the following loci:—the vertices of triangles on the same base, having a given vertical angle.

14. Of the point of intersection of the lines which bisect the angles at the base of such triangles.

(Prove that the angle between each pair of intersectors is the same.)

15. Of a point at which a given straight line subtends a given angle.

16. Of the points of bisection of parallel chords in a circle.

17. Of the points of bisection of equal chords in a circle.

18. Of points from which the tangent to a given circle has a given length.

19. Of the vertices of right-angled triangles on a given base.

20. Of the centres of all circles which touch a given line in a given point.

21. Of the centres of circles which touch a given circle in a given point.

22. Of the centres of circles of given radius which pass through a given point.

23. Of the middle point of a line drawn from a given point to meet a given circle.

24. Shew that the inscribed equilateral triangle is one-fourth of the circumscribed equilateral triangle.

25. A ladder slips down a wall: find the locus of its middle point.

26. If from two fixed points in the circumference of a circle two lines are drawn to intercept a given arc, the locus of their intersections is a circle.

27. Two chords of a circle which do not bisect each other do not pass through the centre.

28. Circles which cut one another cannot have the same centre.

29. Two shillings are moved in the corner of a box so that each always touches one side, and they touch one another; find the locus of the point of contact.

30. Two circles cut one another, and lines are drawn through the points of section, and terminated by the circumferences; shew that the chords which join the extremities of these lines are parallel.

31. Two equal circles intersect in A and B, and any line BCD is drawn to cut both circles. Prove that

$$AC = AD.$$

32. Two equal circles intersect in A, B; a third circle is drawn, with centre A and any radius less than AB, meeting the circles in C, D, on the same side of AB. Prove that B, C, D lie in one straight line.

33. ACD, ADB are two segments of circles on the same base AB; take any point C on the segment ACB, and join CA, CB, and produce them if necessary to meet ADB in D, E. Shew that the arc DE is constant.

34. If two circles cut each other, and from either point of intersection diameters be drawn, the extremities of these diameters and the other point of intersection shall be in the same straight line.

35. If a straight line be drawn to touch a circle and parallel to a chord, the point of contact will bisect the arc cut off by that chord.

36. Perpendiculars AD, CE are let fall from the angles A, C of the triangle ABC on the opposite sides. Prove that the angle ACE is equal to the angle ADE.

37. Two circles intersect in A, B, and tangents AC, AD are drawn to each circle, meeting circumferences in C, D, prove that BC, BD make equal angles with BA.

38. If one of two intersecting circles pass through the centre of the other, prove that the tangent to the first at the point of intersection, and the common chord, make equal angles with the radius to that point from the centre of the second.

39. Given base, altitude, and vertical angle, construct the triangle.

40. To draw a line from a given point such that the

perpendicular on it from a given point shall have a given length.

41. In a given straight line to find a point at which a given straight line subtends a given angle.

42. Describe a circle to touch a given circle, and touch a given line in a given point.

43. Describe a circle of given radius to touch a given line, and have its centre on another given line.

44. Find a point in a given chord produced of a circle, from which the tangent to the circle shall have a given length.

45. With a given radius describe a circle touching two given circles.

46. Describe a triangle, having given the vertical angle and the segments of the base made by the line bisecting the vertical angle.

47. Given base, altitude, and radius of circumscribed circle, construct the triangle.

48. The triangle contained by the two tangents to a circle from any point and any other tangent that meet them has its perimeter double of either of the two tangents. Prove this; and apply it to construct a triangle, having given the vertical angle, perimeter, and altitude.

49. Given the perimeter, the vertical angle, and the line bisecting the vertical angle, construct the triangle.

50. If two of the external angles of a triangle and one internal angle are bisected, prove that the three bisectors intersect in one point.

51. The three perpendiculars to the sides of a triangle drawn through their middle points meet in one point.

52. The three lines which join the angles of a triangle to the middle points of the opposite sides intersect in one point.

53. If two circles touch one another, the lines which join the extremities of parallel diameters towards opposite parts will intersect in the point of contact.

54. The circles described on the sides of a triangle as diameters intersect in the sides, or sides produced, of the triangle.

55. Equilateral triangles are described on the sides of a triangle; prove that the circles described about those triangles pass through one point.

56. If tangents be drawn at the extremities of any two diameters of a circle, the straight lines joining the opposite points of intersection will both pass through the centre.

57. The four common tangents to two circles which do not meet one another intersect, two and two, on the straight line which joins the centres of the circles.

58. Given the altitude, the bisector of the vertical angle, and the bisector of the base, to construct the triangle.

59. Draw circles to touch one side of a given triangle and the other two sides produced.

60. Draw a figure of a triangle with its inscribed, circumscribed, and escribed circles.

61. If a triangle is equilateral, shew that the radii of

the inscribed, and circumscribed, and an escribed circle are to one another as 1, 2, 3.

62. If circles are described with the vertices of a triangle as centres, and so as to pass through the points of contact of the inscribed circle with the adjacent sides, these three circles will touch one another.

63. Place a straight line of given length in a circle so that it shall be parallel to a given diameter of the circle.

64. Place (when possible) a straight line of given length in a circle so that it shall pass through a given point within or without the circle.

65. Given three points describe circles from them as centres so that each may touch the other two.

66. On the side of any triangle equilateral triangles are described externally, and their vertices joined to the opposite vertices of the given triangle, shew that the joining lines pass through one point.

67. O is the centre of the circle inscribed in the triangle ABC, which touches AB, AC in C', B'; if AO cuts the circle in P, and AO produced in P', shew that P, P' are the centres of the inscribed and escribed circles of the triangle $AB'C'$.

68. Shew that a triangle is equal to the rectangle contained by its semi-perimeter and the radius of the inscribed circle.

69. Of all the rectangles inscribable in a circle, shew that a square is the greatest.

70. If a quadrilateral figure can be described about a circle, shew that the sums of its opposite sides are equal. Can a circle be inscribed in (1) a rectangle, (2) a parallelogram, (3) a rhombus?

71. Shew that the inscribed hexagon is three-fourths of the circumscribed hexagon.

72. Shew that the six segments into which the points of contact of the escribed circles of a triangle divide the sides, may be arranged in three pairs of equal segments.

73. Inscribe an octagon in a given circle.

74. Describe a circle (1) to touch three given lines; (2) to intercept equal chords of any given length on three given straight lines.

In how many ways may each of these problems be solved?

75. At any point in the circumference of the circle circumscribing a square, shew that one of the sides subtends an angle three times as great as the others.

76. Find the locus of points at which two sides of a square subtend equal angles.

77. Find the locus of points at which three sides of a square subtend equal angles.

78. If four straight lines intersect one another so as to form four triangles, prove that the four circumscribing circles will pass through one point.

79. Of all triangles inscribable in a circle the greatest is the equilateral. Extend this to the case of a polygon of any number of sides.

80. A straight line is divided into any two parts in C, and ADC, CEB are equilateral triangles on the same side of AB. Find the locus of the intersection of AE and BD.

BOOK III. PROPORTION.

INTRODUCTION.

MEASURES.

A *measure* of a line is any line which is contained in it an exact number of times. Thus an inch is a measure of a foot; and a yard is a measure of a mile. So too the measure of an area is any area which is contained an exact number of times in it. A square inch is thus a measure of a square yard. *A measure is therefore an aliquot part of any magnitude which it measures.* The length of a line, the extent of an area, or any other magnitude, is completely known when we know a measure of it, and how many times it contains that measure.

In measuring any magnitude we take some standard to measure by. Thus in measuring length we take a yard, or a foot, or an inch. In measuring solids we take a cubic inch, a cubic foot, or the like. The standard so taken is called the *unit.* It may be a precise measure of the magnitude measured, or it may not. The number, whether whole or fractional, which expresses how many times a magnitude contains a certain unit is called the *numerical value* of that magnitude in terms of that unit. Thus in speaking of a

line as 7 yards long, a yard is the unit of length, and the numerical value of the line in terms of that unit is 7.

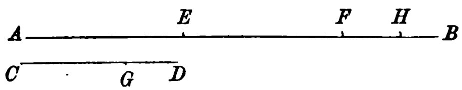
Two lines or magnitudes of the same kind are said to have *a common measure* when there exists a unit of which they can both be expressed as multiples. Thus 15 inches and 1 foot have a common measure, for with the unit 3 inches, their numerical values would be 5 and 4; and with the unit 1 inch their numerical values would be 15 and 12. All whole numbers have unity as a common measure.

The following problem gives a method of finding the greatest common measure of two magnitudes, if any common measure exists.

PROBLEM.

To find the greatest common measure of two magnitudes, if they have a common measure.

Let AB and CD be the two magnitudes. From AB



the greater cut off parts, AE, EF... each equal to CD the less, leaving a remainder FB which is less than CD.

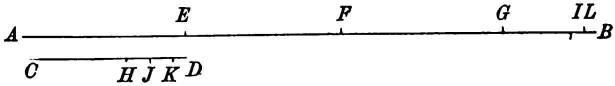
From CD cut off parts, CG..., equal to FB, leaving a remainder GD less than FB.

From FB cut off parts FH, HB... equal to GD: and continue this process until a remainder GD is found which is contained *an exact number* of times in the previous remainder, so that no further remainder is left. The last remainder is then the greatest common measure.

For, firstly, since GD measures FB, it also measures CG; and therefore measures CD. But $CD = AE$ and EF; and therefore GD measures AE, EF and FB; that is it measures AB. Hence GD is *a* common measure of AB and CD.

And again, since every measure of CD and AB must measure AF, it must measure FB or CG, and therefore also GD: hence the common measure cannot be greater than GD; that is GD is the *greatest* common measure.

So also, in the figure adjoining, the first remainder is



GB; the second HD; the third IB; the fourth KD, which is contained exactly twice in IB. Hence KD is the greatest common measure, and it will be seen to be contained twice in IB, and therefore five times in HD, seven times in GB, 12 times in CD, and 43 times in AB.

Hence AB and CD have as their numerical values 43 and 12 in terms of the unit KD.

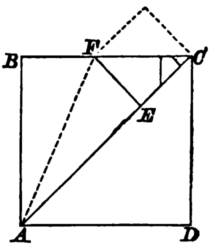
Cor. Every measure of KD is a common measure of AB and CD.

When magnitudes have a common measure they are called *commensurable*. But it is very frequently the case in Geometrical figures, that lines and other magnitudes have no common measure; the process above given continuing indefinitely; the remainder becoming smaller at each step of the process but never actually disappearing. In this case the lines are said to be *incommensurable*.

The following theorem will serve to illustrate incommensurable magnitudes.

THEOREM I.

To prove that the side and diagonal of a square are incommensurable.



Let $ABCD$ be a square: AC the diagonal.

Then will AC and AB be incommensurable.

For since AC is $> AB$ and $<$ twice AB, cut off a part $AE = AB$. Then the common measure of AC and AB, is also a common measure of EC and AB, or of EC and BC.

Draw EF at right angles to AC to meet BC in F: and join AF. Then in the right-angled triangles ABF, AEF the hypothenuse and one side $AB =$ the hypothenuse and one side AE, and therefore $BF = FE$.

And in the triangle FEC, since E is a right angle, and ECF half a right angle, therefore EFC is half a right angle, and therefore $FE = EC$. Therefore $EC = BF$.

Hence the common measure of EC and BC is also a common measure of EC and FC.

But EC and FC are the side and diagonal of a square smaller than $ABCD$; and similarly the common measure of EC and FC can be shewn to be also the common measure of the side and diagonal of a still smaller square; and so on, till the side and diagonal become as small as we please.

Hence it is evident that we shall never find a common measure of AC and BC. For the original relation of side and diagonal is perpetually reproduced, and we are perpetually brought back to the problem with which we started.

The remainder becomes smaller at every step but never actually disappears. There is therefore no common measure of AC and BC; that is, they are incommensurable.

Ratio.

When two magnitudes of the same kind are considered we can compare them with one another, and form a notion of their relative magnitudes: we do this instinctively, and antecedently to all geometrical teaching. To express the notion thus formed we must ascertain how many times the one contains the other, or some aliquot part of the other. Thus if one line contains another 7 times, the one line has the same relative magnitude to the other that the number 7 has to the number 1; and if one line contains the 5th part of another 8 times, the one has to the other the same relative magnitude that 8 has to 5.

This relation of magnitude is called ratio.

It follows from the notion of relative magnitude that the ratio of two magnitudes is the same as that of their numerical values in terms of the same unit. And if two magnitudes have the same numerical values in terms of different units, their ratio is the same as that of their units.

Thus the ratios of the lines whose Greatest Common Measures were ascertained above, are the ratios of 12 to 5 and 43 to 12 respectively. Again the ratio of 7 weeks to 7 days is that of a week to a day.

The fact is that all number is but a kind of ratio.

We are taught in arithmetic to distinguish between *concrete* and *abstract quantities.* Concrete quantities are really existing things: abstract quantities are the means that we use to express the concrete. Seven shillings, five horses, three acres are concrete quantities: seven, five, three

are abstract quantities. The difference is most clearly seen in multiplication. You cannot multiply by a concrete quantity; a multiplier must be abstract. You cannot multiply seven shillings by six shillings, nor five days by three weeks, nor twelve miles by ten cats. Now *abstract quantities* and *ratios* are precisely the same things. We use the phrase "*abstract quantity*" when we are thinking of a quantity in itself; we use the word "*ratio*" when we are thinking of the relation of one quantity to another. The number seven considered in itself is called an abstract quantity, considered as expressing the relation of a week to a day, of a guinea to three shillings, of the number 42 to the number 6, it is called a ratio.

All numbers, therefore, as we said before, are ratios. But there is an important distinction between numbers and ratios; and though all numbers are ratios, all ratios are not numbers.

Numbers are essentially discontinuous, and therefore unsuited immediately to express the relations of magnitudes that change continuously. If a line 3 inches long is conceived as stretched till it becomes 4 inches long it passes continuously from one length to the other, but the arithmetical expression of the change is discontinuous, however small a unit we take: if we take $\frac{1}{1000}^{\text{th}}$ of an inch as the unit, the line is at first 3000, then 3001, 3002,...and finally 4000 of these units, but we cannot arithmetically express all the values the line takes in passing from 3000 to 3001; that is, all the ratios which its length has to $\frac{1}{1000}^{\text{th}}$ of an inch.

Here lies the fundamental difficulty in the application of arithmetic to Geometry. Arithmetic deals with numbers which are discontinuous. Geometry with ratios which are

continuous and only coincide with numbers at regular intervals. We may make those intervals as small as we please, but we cannot get rid of them.

Commensurable magnitudes have to one another a ratio which can be expressed in numbers. For if they contain their common measure m times and n times respectively, they have the same ratio as m to n.

This is generally called the ratio of m to n, which is written $m : n$ or $\frac{m}{n} : 1$ or more simply $\frac{m}{n}$; and if A, C are the magnitudes it is said that $\frac{A}{C} = \frac{m}{n}$ or that $A = \frac{m}{n} C$.

Incommensurable magnitudes have not to one another the ratio of any two numbers however great. For if they had the ratio of $p : q$, and the first were divided into p equal parts, the second would contain q of those parts; and thus they would have a common measure, viz. one of these parts.

But numbers can be found which express within any specified degree of accuracy the ratio of incommensurable magnitudes.

For if A and B be the magnitudes, and B is divided into n equal parts, there must be some number m, such that A contains m but not $m + 1$ such parts; therefore the ratio of $A : B$ is greater than $\frac{m}{n}$, but less than $\frac{m+1}{n}$; and these ratios differ only by $\frac{1}{n}^{\text{th}}$, which by increasing n may be made as small as we please.

This conclusion may be differently expressed thus; that if A and B are incommensurable, they may be made commensurable by adding to either of them a magnitude less

than one which shall be as small as we please. For by adding to A a quantity less than $\frac{1}{n}^{\text{th}}$ of B, A thus increased would contain the n^{th} part of B $\overline{m+1}$ times exactly.

It follows from what has been said that the ratio of two magnitudes is the quantity of one in terms of the other.

It is important to observe that since magnitude and ratio are both continuous, there is always a magnitude that has a given ratio to any given magnitude.

Compound Ratio.

When a magnitude is altered successively in two or more ratios it has to the final result a ratio which is compounded of the given ratios.

Thus if $m : n$, $p : q$ be two ratios, and a magnitude A is first altered in the ratio $m : n$, and then the result altered in the ratio $p : q$, and the final result thus obtained is B, then the ratio $A : B$ is compounded of the ratio $m : n$, $p : q$; for the single alteration indicated by ratio $A : B$ produces the joint effect of the two alterations indicated by the ratio $m : n$, $p : q$.

This process is called the composition of ratios. It is plainly not necessary to suppose that A has been actually altered till it has become B. It is enough if B is what A would have become if so altered.

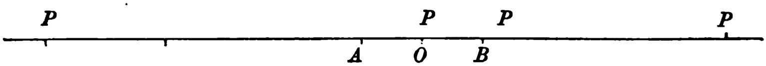
When two equal ratios are compounded the ratio resulting from the composition is said to be the duplicate of the original ratio. When three equal ratios, the triplicate; and so on.

It is evident that different ratios cannot have the same duplicate or triplicate ratio.

The following theorem furnishes a valuable exercise on ratios.

THEOREM 2.

If A *and* B *be two fixed points in a straight line of indefinite length, and* P *a moveable point in that line, then the ratio of* PA *to* PB *may have any value, from* 0 *to infinity, and there are two and only two positions of* P *such that* PA : PB = *any given ratio.*



The value infinity is represented by the symbol ∞.

Let O be the point of bisection of AB; then if P is at O the ratio $\frac{PA}{PB} = 1$.

Conceive the point P to move to the right towards B, then the ratio $\frac{PA}{PB}$ continually increases until, when P approaches indefinitely near to B the ratio becomes infinite; and for intermediate positions it has passed continuously through every value between 1 and ∞.

When P is at the right of B the ratio

$$\frac{PA}{PB} = \frac{PB + AB}{PB} = 1 + \frac{AB}{PB},$$

and is therefore greater than 1.

When PB is very small $\frac{AB}{PB}$ is very large, and as PB increases $\frac{AB}{PB}$ diminishes until it becomes indefinitely small,

and therefore $\frac{PA}{PB}$ becomes as nearly equal to 1 as we please, and has passed continuously through every value between ∞ and 1.

Hence for any assigned value of the ratio greater than 1 there are two positions for P, one between O and B, and one to the right of B.

Similarly as P moves from O to A, $\frac{PA}{PB}$ passes through every value from 1 to 0, and as it moves to the left of A it passes through every value from 0 to 1, and therefore for every value of the ratio less than 1 there are two positions for P, one between O and A, and one to the left of A.

When P is between A and B it is said to divide AB *internally;* when not between A and B it divides AB *externally*, and PA, PB are still spoken of as the segments of the line AB.

Proportion.

Def. *Proportion* consists in the equality of ratios.

Four magnitudes are called *proportionals*, or are in proportion, when the 1st has the same ratio to the 2nd that the 3rd has to the 4th.

If A, B, C, D are the magnitudes in proportion, of which A and B are of the same kind, and C and D of the same kind, this is expressed by the notation $A : B :: C : D$, or by $\frac{A}{B} = \frac{C}{D}$. A and D are called the extremes, and B and C the means.

Def. The 1st and 3rd terms are called *homologous*, as

occupying the same place as antecedent in the ratio; so also are the 2nd and 4th, as consequent.

The ratios of commensurable magnitudes may be expressed numerically by their numerical values in terms of their common measure; and in such cases it is easy to ascertain whether the proportion is true. But if the magnitudes are incommensurable, their numerical values cannot be *exactly* expressed, and yet the ratios may be *exactly* equal, as is shewn in the following Theorem.

THEOREM 3.

If A, B, C, D *be four magnitudes such that* B *and* D *always contain the same aliquot part of* A *and* C *respectively the same number of times, however great the number of parts into which* A *and* C *are divided, then* A : B :: C : D.

For suppose A and B to be incommensurable magni-

tudes, and when A is divided into n equal parts, let B contain m, but not $m+1$ of these parts. And let C and D be two other magnitudes, such that when C is divided into n equal parts, D contains m but not $m+1$ of these parts.

If D' is the 4th proportional to A, B, C, then D' also must contain m but not $(m+1)$ of these parts.

Now if D' differed from D by any quantity however small, it would be possible by sufficiently increasing n to make

the n^{th} part of C still smaller than this quantity. And then when D contained this n^{th} part of C m times but not $m+1$ times, D' would contain it either $m-1$ or $m+1$ times, according as D' was less or greater than D. But then D' would plainly not be the fourth proportional to A, B, and C. Therefore D' does not differ from D, that is $D=D'$ and the four magnitudes A, B, C, D are proportionals.

This reasoning is frequently applicable in Geometry to prove that four magnitudes are proportionals, and is applicable alike to commensurable and incommensurable magnitudes.

COR. 1. Permutando. *If* A, B, C, D *are magnitudes of the same kind, and* A : B :: C : D, *then is* A : C :: B : D.

For let D' be the 4^{th} proportional to A, C, B, so that $A : C :: B : D'$. Then the ratio of A to C is that of the n^{th} part of A to the n^{th} part of C, into which parts they were divided: and since B must have to D' the same ratio, B and D' cannot contain a different number of these n^{th} parts respectively; so that since B contains m and not $m+1$ of the n^{th} parts of A, D' must contain m and not $m+1$ of the n^{th} parts of C. Now, precisely as before, if D' differed from D by any quantity however small, it might be shewn, by increasing n, to contain the n^{th} part of C either $m-1$ or $m+1$ times, which is impossible. Therefore $D'=D$ and $A : C :: B : D$.

COR. 2. Invertendo. If A, B, C, D are in proportion

$$B : A :: D : C.$$

COR. 3. Componendo. Also $A+B : B :: C+D :: D$; and Dividendo. $A-B : B :: C-D : D$.

COR. 4. Ex æquali. If $A : B :: C : D$ and $B : E :: D : F$; then $A : E :: C : F$.

For $A : E$ is compounded of $A : B$ and $B : E$, and $C : F$ is compounded of the ratio $C : D$ and $D : F$ which are respectively equal to the ratios $A : B$ and $B : E$.

COR. 5. Addendo. If $A : B :: A' : B' :: A'' : B''$;

then $$A + A' + A'' : B + B' + B'' :: A : B.$$

These and many other proportions are derivable from the original proportion $A : B :: C : D$, by the method of proof given above.

Remark. All these derived proportions may be obtained by the following method:—

(1) If A and B are commensurable, and therefore also C and D, let them have as their numerical values a, b, c, d respectively.

Then since $\quad A : B :: C : D,$

therefore $\quad \frac{a}{b} = \frac{c}{d}$ and $\therefore \frac{a}{c} = \frac{b}{d}$,

and therefore $\quad A : C :: B : D.$

(2) If A and B are incommensurable, and therefore also C and D, let B' and D' be taken differing from B and D by quantities less than $\frac{1}{n}^{\text{th}}$ of A and $\frac{1}{n}^{\text{th}}$ of C respectively (see p. 52), so that $A : B' :: C : D'$.

Then by the first case $A : C :: B' : D'$.

And since B' and D' differ from B and D by quantities which may be made as small as we please, we infer that

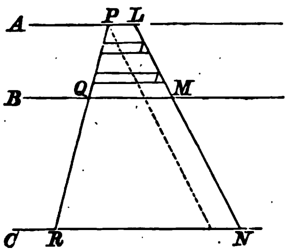
$$A : C :: B : D.$$

SECTION I.

Application of Proportion to Lines.

Theorem 4.

If two straight lines are cut by three parallel straight lines, the segments made on the one are in the same ratio as the segments made on the other.



Let A, B, C be the three parallels, PQR, LMN any two lines intersected by them; then shall

$$PQ : QR :: LM : MN.$$

For if PQ be divided into any number of equal parts, and through the points of division lines be drawn parallel to A or B, LM will be divided by those lines into the same

number of equal parts, as may be proved by the methods of Book I. And if from QR parts equal to those of PQ are cut off, and lines drawn through the points of division, parallel to B or C, it is clear that MN will contain the same number of parts each equal to those of LM, that QR contains of those of PQ; that is, QR and MN will contain the same aliquot parts of PQ and LM the same number of times, however great the number of parts into which PQ and LM are divided.

Hence $\qquad PQ : QR :: LM : MN.$

COR. 1. It follows from the same reasoning that

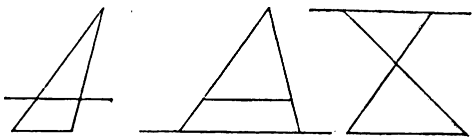
$$PQ : PR :: LM : LN,$$

and that $\qquad QR : PR :: MN : LN,$

and that $\qquad PQ : LM :: QR : MN,$

(by TH. 3. CORS. 1, 2, 3).

COR. 2. *If a line be drawn parallel to one side of a triangle, it will cut the other sides or the other sides produced proportionally.*



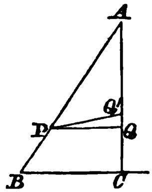
COR. 3. The converse of this corollary is true. That is *if a line cuts two sides of a triangle, both internally or both externally, proportionally, the line shall be parallel to the base of the triangle.*

For let $AP : PB :: AQ : QC$, and suppose PQ not parallel to BC, but let PQ' be parallel if possible;

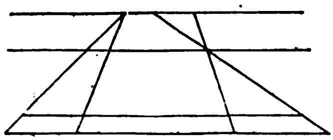
then $AP : PB :: AQ' : Q'C$,

and therefore $AQ : QC :: AQ' : Q'C$,

which is impossible.

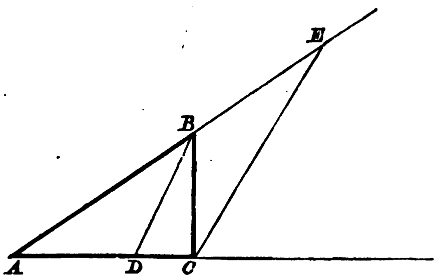


COR. 4. *If any number of lines be parallel and be cut by other lines, the intercepts made on the latter by the parallels will be proportionals.*



THEOREM 5.

If a line bisect the vertical angle of a triangle and meet the base, it will divide the base into two segments which have to one another the ratio of the sides of the triangle.



Let ABC be a triangle, BD the bisector of the angle ABC.

Then will $AD : DC :: AB : BC$.

Draw CE parallel to BD to meet AB produced.

Then by parallelism the angle $BCE =$ the angle DBC, and the angle $BEC =$ the angle ABD. But $ABD = DBC$, and therefore the angle $BCE =$ the angle BEC; and therefore $BE = BC$.

But because AE, AC are cut by the parallels DB, CE;

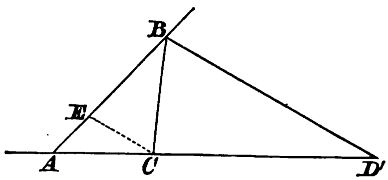
therefore $AD : DC :: AB : BE$,

that is, $AD : DC :: AB : BC$.

COR. 1. *Conversely, if $AD : DC :: AB : BC$, then BD is the bisector of the angle ABC.*

For there is only one internal bisector of the angle, and only one point D which divides the base internally, so that

$$AD : DC :: AB : BC.$$



COR. 2. *If BD' bisects the exterior angle, D', then also*

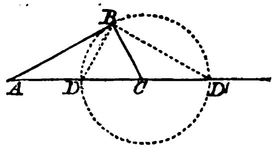
$$AD' : D'C :: AB : BC.$$

Draw CE parallel to BD', as before: and apply the same method of proof.

COR. 3. *If* AB = BC, *then the ratio of* AD′ : D′C *becomes* = 1, *which indicates that* D′ *is at an infinite distance (by Theorem 2). Hence the external bisector of the vertical angle of an isosceles triangle is parallel to the base.*

COR. 4. *If* B *moves so that the ratio* AB : BC *is constant, the bisectors of the interior and exterior angles will always pass through the fixed points* D, D′ *which divide* AC *internally and externally in that ratio.*

COR. 5. *Hence the locus of* B *is a circle described on* DD′ *as diameter. For the angle* DBD′ *is a right angle; and therefore the semicircle on* DD′ *will always pass through* B.

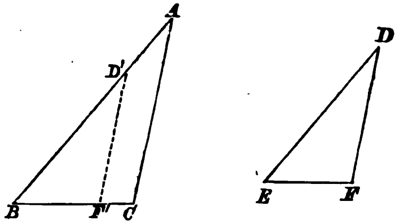


Def. Similar figures are such that the angles of the one are respectively equal to the angles of the other, and have the sides about the equal angles proportionals.

THEOREM 6.

If two triangles have two angles of the one equal respectively to two angles of the other, the triangles shall be similar, the sides which are opposite the equal angles being homologous.

Let *ABC, DEF* be the two triangles, which have two



angles of the one equal to two angles of the other, and therefore have also their remaining angles equal.

Then shall they be similar, that is

$$AB : BC :: DE : EF,$$

and $$BC : CA :: EF : FD,$$

and $$CA : AB :: FD : DE.$$

Conceive the angle E placed on the angle B; then F and D would fall as F' and D' on BC and BA, or on those lines produced: and because the $\angle F =$ the $\angle C$, therefore $F'D'$ is parallel to CA;

and therefore $BF' : BC :: BD' : BA$,

and therefore $BF' : BD' :: BC : BA$,

that is $EF : ED :: BC : BA$.

Similarly by placing F on C, and D on A, the other proportions are obtained; and therefore the triangles are similar.

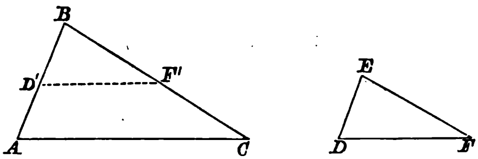
This theorem is a generalization of Theorem 16 in Book I. *If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another triangle, these triangles will be equal in all respects.*

Hence it will be observed that the equality of the angles involves the similarity of the triangles; and the additional equality of a pair of corresponding sides involves the identity of the triangles.

THEOREM 7.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, then will the triangles be similar.

Let the triangles ABC, DEF have the angles at B and



E equal, and let $BA : BC :: ED : EF$, then will the triangles be similar.

Conceive the angle E placed on the equal angle B, then D and F will fall as at D' and F' on the sides BA, BC,

and since $\quad BA : BC :: ED : EF$,

therefore $\quad BA : BD' :: BC : BF'$,

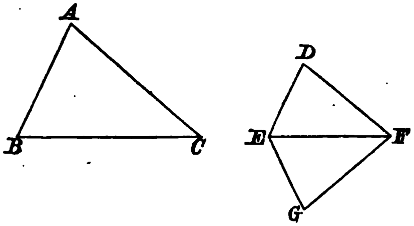
and therefore $D'F'$ is parallel to AC, TH. 4. COR. 3. and the angles $BD'F'$ and $BF'D'$, that is, D and F, are equal respectively to the angles A and C. Hence the triangles are equiangular and therefore similar.

It will be observed that this theorem is a generalization of Book I. Theorem 17. *If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another, the triangles will be equal in all respects.*

THEOREM 8.

If the sides about each of the angles of two triangles are proportionals, the triangles will be similar.

Let ABC, DEF be two triangles which have their sides



about each of their angles proportional,

that is, $AB : BC :: DE : EF$,

and $BC : CA :: EF : FD$,

and $CA : AB :: FD : DE$.

Conceive a triangle equiangular to ABC applied to EF, on the opposite side of the base EF, so that the angles FEG, EFG are equal to B and C respectively.

Then the triangle GEF is equiangular to ABC, and therefore similar to it,

and therefore $GE : EF :: AB : BC$,

but $AB : BC :: DE : EF$,

and therefore $GE : EF :: DE : EF$,

and therefore $GE = ED$.

Similarly $GF = DF$,

and the triangle DEF is therefore equiangular to GEF, and therefore also to ABC.

Therefore the triangle DEF is similar to the triangle ABC.

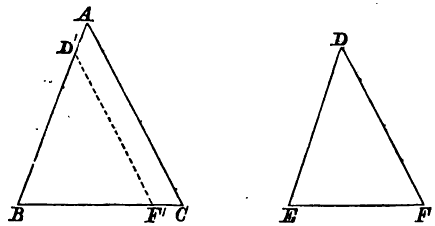
This theorem is a generalization of Book I. Theorem 18. *If the three sides of one triangle are respectively equal to the three sides of another, these triangles will be equal in all respects.*

THEOREM 9.

If two triangles have the sides about an angle of the one triangle proportional to the sides about an angle of the other, and have also the angle opposite that which is not the less of the two sides of the one equal to the corresponding angle of the other, these triangles will be similar.

Let ABC, DEF be the two triangles, in which

$$BA : AC :: ED : DF,$$



and let AC be not less than AB, and DF therefore not less than DE, and also let the angle B = the angle E.
Then shall the triangles be similar.

Cut off $BD' = ED$, and draw to BC an oblique $D'F'$ parallel to AC.

Then by similar triangles $BD'F'$ and BAC,

$$BD' : D'F' :: BA : AC,$$

that is, $$ED : D'F' :: BA : AC,$$

but $$ED : DF :: BA : AC,$$

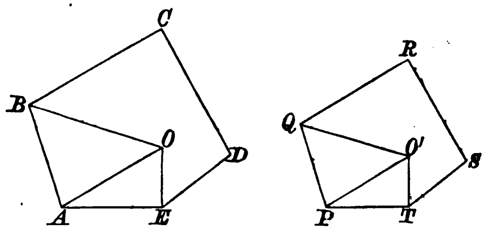
therefore $$DF = D'F',$$

and since DF is not less than DE, the two triangles $BD'F'$, EDF are equal in all respects by Book I. Theorem 19, that is, the triangle EDF is equiangular to the triangle $BD'F'$, and therefore also the triangle EDF is equiangular to ABC, and therefore similar to ABC.

This theorem is a generalization of Book I. Theorem 19.

THEOREM 10.

Similar polygons can be divided into the same number of similar triangles.



Let $ABCDE$, $PQRST$ be similar polygons, that is, let the angles $A, B, C\ldots$ of the one equal the angles $P, Q, R\ldots$ of the other respectively, and let the sides which contain the equal angles be proportional.

That is, let $EA : AB :: TP : PQ$, and similarly for the sides about the other angles.

Take any point O within the polygon $ABCDE$. Join OA, OE. Make the angles TPO', PTO' equal to EAO, AEO respectively.

Then if OB, OC... $O'Q$, $O'R$... be drawn, the polygons will be divided into the same number of similar triangles.

For join OB, $O'Q$.

Then since the triangles OAE, $O'PT$ are equiangular,

$$\therefore \quad OA : AE :: O'P : PT;$$

but

$$AE : AB :: PT : PQ;$$

$$\therefore \quad OA : AB :: O'P : PQ,$$

and since the angle BAE = the angle QPT, and $OAE = O'PT$,

$$\therefore \quad \text{the angle } OAB = \text{the angle } O'PQ;$$

$$\therefore \quad \text{the triangles } OAB,\ O'PQ \text{ are similar};$$

and therefore $OBA = O'QP$, and $OB : BA :: O'Q : PQ$; in the same way it may be shewn that if OC, $O'R$ are joined, the triangle OBC is similar to the triangle $O'QR$.

Hence the polygons can be divided into the same number of similar triangles.

The points O, O' are then called homologous points, and the lines OB, $O'Q$; OA, $O'P$, &c., homologous lines.

Cor. 1. *The perimeters of similar polygons have to one another the ratio of the homologous sides of the polygon.*

For since $\dfrac{AE}{PT} = \dfrac{AB}{PQ} = \dfrac{BC}{QR} = \ldots$ therefore also by

(Theorem 3. Cor. 5) $\dfrac{AE + AB + BC + \ldots}{PT + PQ + QR + \ldots} = \dfrac{AE}{PT}$.

COR. 2. *The perimeters of similar polygons have to one another the ratio of any pair of homologous lines.*

COR. 3. *It follows further that if two polygons have the same number of sides, and the diagonals from the corresponding angular points are drawn, and are such that*

(1) *All the angles of one figure are equal to the corresponding angles of the other figures, or* (2) *the lines of one figure are proportional to the corresponding lines of the other figure,*

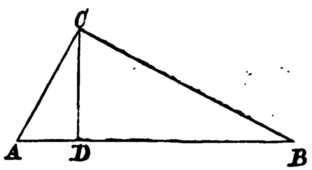
Then the two figures are similar.

These theorems like the corresponding theorems in Book I. may be used to establish many geometrical properties. We shall give some examples of their application.

EXAMPLES.

EX. 1. *In a right-angled triangle the perpendicular from the right angle on the hypothenuse divides the triangle into two others which are similar to the whole and to one another.*

Let ACB be the triangle, right-angled at C; CD the perpendicular.



Then the triangles CAD, BAC have two angles CAD and CDA of the one equal respectively to BAC, BCA of the other; therefore they are equiangular, and similar.

In the same manner DCB is equiangular and similar to either DAC or CAB.

COR. $$AD : DC :: DC : DB,$$

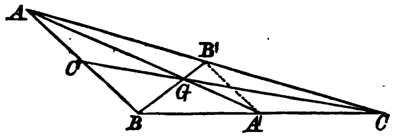
or the perpendicular from the right angle of a right-angled triangle on the hypothenuse is a mean proportional between the segments of the base.

Also $$BA : AC :: AC : AD,$$

and $$AB : BC :: BC : BD,$$

or the side of a right-angled triangle is a mean proportional between the hypothenuse and the projection on it of that side.

EX. 2. *The three lines drawn from the angular points of a triangle to bisect the opposite sides will intersect in one point.*



Let ABC be the triangle, A', B', C' the middle points of its sides. Then AA', BB', CC' will pass through one point.

Join $B'A'$. Then since CA, CB are bisected in B', A',

$$\therefore\ CB' : B'A :: CA' : A'B,$$

and therefore $B'A'$ is parallel to AB,

and because the triangles CAB, $CB'A'$ are similar,

therefore $\qquad AB : B'A' :: AC : B'C$;

but $AC = 2B'C$; and therefore $AB = 2B'A'$.

Now let AA' and BB' cut in G.

Then the triangles GAB, $GA'B'$ are equiangular,

and therefore $\qquad AG : GA' :: AB : B'A'$;

but $\qquad AB = 2B'A'$, and therefore $AG = 2GA'$,

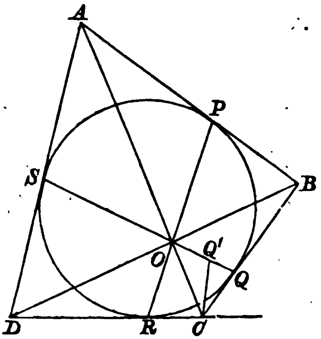
or $\qquad GA' = \frac{1}{3}AA'$.

In the same manner it may be shewn that CC' cuts AA' at a point distant $\frac{1}{3}AA'$ from A'; that is, CC' passes through G the intersection of AA' and BB'.

Ex. 3. *If a quadrilateral figure be described about a circle, and the points of contact of opposite sides be joined, prove that these lines and the diagonals of the quadrilateral figure all intersect in one point.*

Let $ABCD$ circumscribe the circle, P, Q, R, S being the points of contact. Let AC cut SQ in O, draw CQ' parallel to AS.

Since SQ is the chord of contact of the tangents AS, BQ, the angle ASO = the angle BQO.



Therefore CQO is supplementary to ASO, or $OQ'C$,

and therefore $CQO = CQ'Q$, and $CQ = CQ'$.

But from the triangles ASO, COQ', which are similar, we get

$$AO : CO :: AS : CQ' \text{ or } AS : CQ.$$

Similarly, if PR intersected AC in O', $AO' : CO' :: AP : CR$; but $AP : CR :: AS : CQ$, and therefore $AO : CO :: AO' : CO'$, therefore SQ and PR divide AC in the same ratio, or O and O' are the same points. Hence AC passes through the intersection of SQ and PR.

Similarly BD passes through the intersection of SQ and PR;

therefore the four lines AC, BD, PR, SQ pass through one point.

SECTION II.

AREAS.

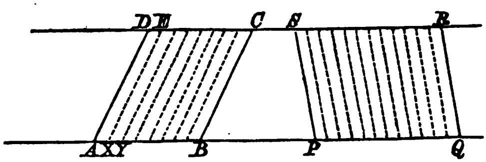
We have seen how *lines* are represented by *numbers;* a line being known when it contains a known unit a known number of times. We now proceed to the consideration of *areas* and their numerical representation.

The fundamental theorem is the following:

THEOREM II.

Parallelograms of the same altitude are to one another as their bases.

Let $ABCD$, $PQRS$ be parallelograms of the same alti-



tude on the bases AB, PQ.

Then shall $DABC$ be to $SPQR$ as AB to PQ.

Divide AB into any number of parts AX, XY... and from PQ cut off parts equal to the parts of AB; and through X, Y... the points of section let lines be drawn parallel to the sides of the parallelogram.

Then the parallelograms DX, EY... into which $DABC$ is divided are all equal to one another, and equal to those cut off from the parallelogram $SPQR$, since they are on equal bases and of the same altitude; and therefore DX is the same aliquot part of $DABC$ that AX is of AB.

Hence the parallelogram $SPQR$ contains any aliquot part of $DABC$ as many times as PQ contains the same aliquot part of AB, into however many parts AB and DB are divided.

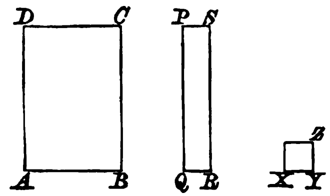
Therefore $AB : PQ :: DABC : SPQR$.

COR. 1. *Triangles of the same altitude are to one another as their bases.*

For a triangle is half the parallelogram on the same base and having the same altitude as the triangle.

COR. 2. *If the unit of area be the square on the unit of length, then will the numerical value of a rectangle be the product of the numerical values of its base and altitude.*

For let $ABCD$ be a rectangle, XY the unit of length,



and XZ the square on XY the unit of area. And let AB contain XY m times, and AD contain it n times.

Then will the numerical value of $ABCD$ be mn.

For construct a rectangle $PQRS$ on a base equal to XY, and with altitude equal to AD.

Then $\qquad DABC : PQRS :: AB : QR,$

$$:: m : 1,$$

that is, $\qquad DABC = m \, . \, PQRS,$

and $\qquad PQRS : ZX :: PQ : XY,$

$$:: n : 1,$$

that is, $\qquad PQRS = n \, . \, ZX;$

and therefore $\qquad DABC = mn \, . \, ZX,$

that is, if ZX is the unit, the numerical value of $DABC$ will be mn.

This is generally expressed by saying that the area of a rectangle is the product of its base and altitude.

Remark. If the rectangle is a square, whose side is m, the area will be m^2: hence if AB is a line, or its numerical value, AB^2 represents either the geometrical square on the line, or the arithmetical square of the number which is the value of AB. And the rectangle contained by AB and CD may be written $AB \times CD$; for the product of the numbers AB and CD is the number which represents the rectangle, on the understanding that the unit of area is the square of the unit of length.

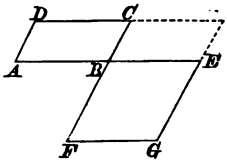
COR. 3. *The area of any parallelogram is the product of its base into its altitude.*

COR. 4. *The area of any triangle is half the product of its base and its altitude.*

COR. 5. *Hence the area of any rectilineal figure can be found by dividing it into triangles, and finding the areas of the triangles.*

THEOREM 12.

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.



Let $ABCD$, $EBFG$ be the parallelograms, and let them be placed so as to have AB, BE in one straight line, and therefore also, since the parallelograms are equiangular, so as to have CB, BF in one straight line.

Complete the parallelogram CBE.

Then the ratio of $DB : BG$ is compounded of the ratios of $DB : CE$ and of CE to BG.

But $$DB : CE :: AB : BE,$$

and $$CE : BG :: CB : BF;$$

therefore, the ratio of $DB : BG$ is compounded of the ratios $AB : BE$ and $CB : BF$.

COR. 1. *Triangles which have one angle of the one equal to one angle of the other, are to one another in the ratio compounded of the ratios of the sides containing that angle.*

COR. 2. *Equal parallelograms which are also equiangular, have their sides reciprocally proportional.*

For, in the figure above, the ratio compounded of the ratio $AB : BE$ and $CB : BF$ must be unity; and therefore $AB : BE :: BF : CB$, or the sides of the parallelograms are reciprocally proportional.

COR. 3. *Conversely, equiangular parallelograms which have their sides reciprocally proportional, are equal.*

COR. 4. *Hence, if four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means, and conversely.*

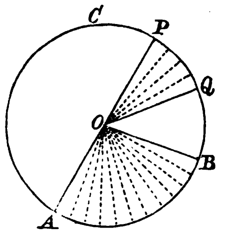
COR. 5. *If three straight lines are in continued proportion, that is, if the first is to the second, as the second to the third, then will the rectangle contained by the extremes be equal to the square on the mean, and conversely.*

COR. 6. *The ratio of two parallelograms or two triangles is the ratio compounded of the ratios of their bases and altitudes.*

THEOREM 13.

In any circle angles at the centre have to one another the ratio of the arcs on which they stand, or of the sectors which they include.

Let ABC be a circle, of which O is the centre.



And let AOB, POQ be two angles at the centre.

Then $\angle AOB : \angle POQ :: \text{arc } AB : \text{arc } PQ,$

$:: \text{sector } AOB : \text{sector } POQ.$

Divide the arc AB into any number of equal parts, and from the arc PQ cut off arcs equal to these. Join the points of division to O.

Then, since equal arcs subtend equal angles at the centre, the arc AB, and the angle AOB, are divided into the same number of parts.

And therefore POQ contains an aliquot part of AOB the same number of times that PQ contains the same aliquot part of AB, into however many parts AB and AOB are divided; therefore

$$\angle AOB : \angle POQ :: \text{arc } AB : \text{arc } PQ.$$

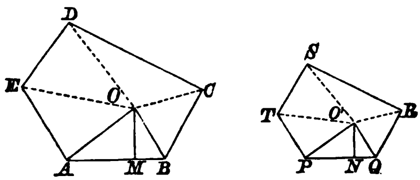
In precisely the same manner, since equal angles contain equal sectors, it may be shewn that

$$\angle AOB : \angle POQ :: \text{sector } AOB : \text{sector } POQ.$$

THEOREM 14.

Similar polygons are to one another in the ratio of the squares of their homologous sides.

Let $ABCDE$, $PQRST$ be similar polygons.



Divide each of them into the same number of similar triangles by lines drawn from the points O, O'.

Let OAB, $O'PQ$ be two similar triangles.

Draw OM, $O'N$ perpendicular to AB, PQ.

Now $$\frac{OAB}{O'PQ}=\frac{AB\times OM}{PQ\times O'N}=\frac{AB}{PQ}\times\frac{OM}{O'N};$$

but by similar triangles OAM, $O'PN$ and OAB, $O'PQ$ we have

$$\frac{OM}{O'N}=\frac{OA}{O'P}=\frac{AB}{PQ}.$$

Therefore, substituting $\frac{AB}{PQ}$ for $\frac{OM}{O'N}$ above, we have

$$\frac{OAB}{O'PQ}=\frac{AB}{PQ}\times\frac{AB}{PQ}$$
$$=\frac{AB^2}{PQ^2}.$$

In the same manner it may be shewn that

$$\frac{OAE}{O'PT}=\frac{AE^2}{TP^2}=\frac{AB^2}{PQ^2},$$

since the polygons are similar, and therefore $\frac{AE}{TP}=\frac{AB}{PQ}$: and similarly for the other triangles;

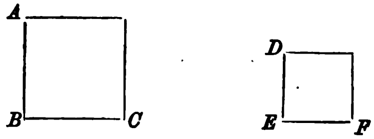
$$\therefore \frac{OAB}{O'PQ}=\frac{OAE}{O'PT}=\dots \quad =\frac{AB^2}{PQ^2}.$$

Therefore the sum of OAB, OAE... is to the sum of $O'PQ$, $O'PT$... as $AB^2 : PQ^2$; that is, the polygons are to one another as the squares of their homologous sides.

COR. 1. *Hence the polygons are to one another as the squares of any homologous lines in them.*

The polygons are said to be *similarly described* on their homologous sides.

Cor. 2. It follows from (Theorem 12) that *the ratio of the squares on two lines is the duplicate ratio of the lines.* For if ABC, DEF are squares, then $AC : DF$ is the ratio



compounded of the ratios $AB : DE$ and $BC : EF$, that is, of $BC : EF$, and $BC : EF$; which is defined to be the duplicate ratio of $BC : EF$.

Moreover, if three straight lines are in continued proportion, the 1st : 3rd in the duplicate ratio of the 1st : 2nd.

Cor. 3. *Therefore further if three straight lines be in continued proportion, the* 1*st* : 3*rd as any polygon described on the* 1*st* : *the similar and similarly described polygon on the* 2*nd.*

Relation of Algebra to Geometry.

We now begin to see how algebra and arithmetic may assist Geometry. For example, we have learnt in algebra that $a^2 - b^2 = (a+b)\ (a-b)$. Now suppose a and b to be *the numerical values of two lines*, then $a+b$ is the numerical value of the sum of the line; $a-b$ is that of their difference.

And therefore $(a+b)(a-b)$ is the numerical value of the rectangle contained by the sum and difference of the two lines.

And a^2-b^2 is the difference of the numerical values of the squares on the two lines. Hence the algebraical identity $a^2-b^2=(a+b)(a-b)$ proves the geometrical fact, (which indeed we have learnt before) that the difference of the squares of two lines is equal to the rectangle contained by the sum and difference of those lines.

So again since $(a+b)^2=a^2+2ab+b^2$ has a geometrical signification which the student should work out for himself in the same manner.

It will be useful for him to work out the geometrical meaning (with figures) of the following identities:

$$(a-b)^2=a^2-2ab+b^2,$$

$$a(a+b)=a^2+ab,$$

$$(a+b)^2=a(a+b)+b(a+b),$$

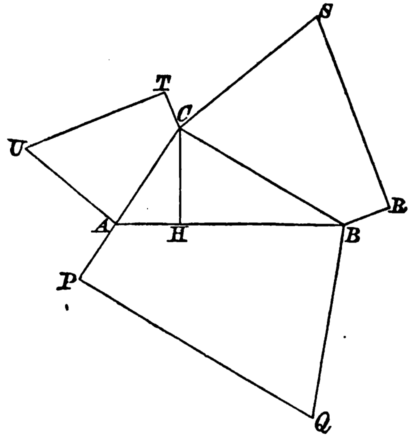
$$(a+b)^2+(a-b)^2=2a^2+2b^2.$$

The following theorems are useful deductions from the preceding.

EXAMPLES.

(1) *In any right-angled triangle any figure described on the hypothenuse is equal to the similar and similarly described figures on the two sides.*

Let ABC be a triangle right-angled at C; and let



$APQB$, $BRSC$, $CTUA$ be similar figures similarly described on the sides AB, BC, CA, that is, figures of which AB, BC, CA are homologous sides.

Then will $\quad APQB = BRSC + CTUA.$

Draw CH perpendicular to AB.

Then $AB : BC :: BC : BH$ by similar triangles, and therefore

$$APQB : BRSC :: AB : BH \text{ (Theorem 14, Cor. 3)},$$

in the same manner it may be shewn that

$$APQB : CTUA :: AB : AH,$$

and therefore

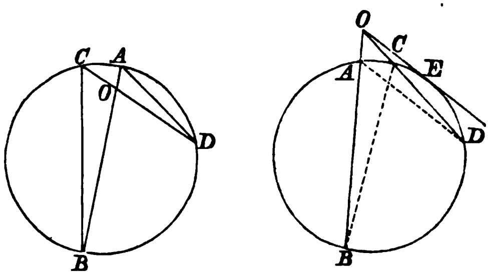
$$APQB : BRSC + CTUA :: AB : BH + AH;$$

but $$AB = BH + AH;$$

and therefore $$APQB = QRSC + CTUA.$$

It is obvious that a special case of this theorem is the theorem proved before, that the square on the hypothenuse of a right-angled tirangle is equal to the sum of the squares on the sides.

(2) *If through any point* O *within or without a circle, chords are drawn, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.*



Let AOB, COD be the chords through O. Then is $AO \times OB = CO \times OD$.

For join CB, AD. Then since the angle D = angle B in the same segment, and the angle at O common to the two triangles AOD, BOC, the triangles are equiangular and similar,

$$\therefore AO : OD :: CO : OB,$$

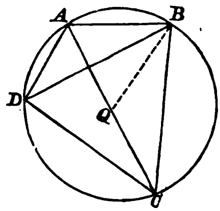
$$\therefore AO \times OB = CO \times OD.$$

COR. If one of the secants OCD, in figure 2, became a tangent, as OE, then OC and OD are equal to OE; and therefore $AO : OE :: OE : OB$, and $OE^2 = AO \times OB$, or *the square on the tangent to a circle from any point is equal to the rectangle contained by the intercepts on the secant drawn from that point.*

(3) *In a quadrilateral inscribed in a circle the rectangle contained by the diagonals is equal to the sum of the rectangles contained by the opposite sides.*

Let $ABCD$ be the quadrilateral inscribed in a circle. Then will

$AC \times BD = AB \times DC + AD \times BC.$



At the point B make the angle CBQ equal to the angle ABD.

Then in the triangles ABD, QBC, since the angle QBC = the angle ABD, and the angle ADB = the angle QCB, therefore the triangles are similar; and therefore $AD : DB :: QC : BC$, and therefore $AD \times BC = DB \times QC$.

In the same manner, since the triangles ABQ, DBC are similar, $AB : QA :: DB : DC$, and therefore

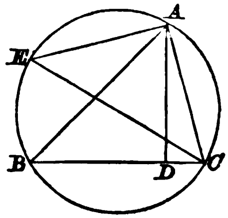
$$AB \times DC = DB \times QA.$$

Therefore $AD \times BC + AB \times DC = DB \times QC + DB \times QA$

$$= DB \times AC.$$

This is known as Ptolemy's theorem.

(4) *In every triangle* ABC, *the rectangle contained by the two sides* AB, AC *is equal to the rectangle contained by the diameter* CE *of the circumscribing circle, and the perpendicular* AD, *let fall on* BC.



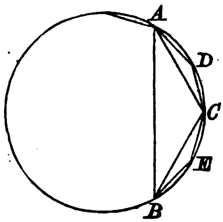
For the triangles ADB, CAE have the angles $ABD = CEA$ and $ADB = CAE$; therefore they are similar, and therefore

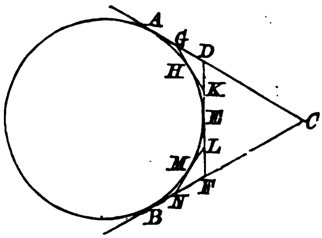
$$AB : AD :: CE : AC,$$
$$\therefore AB \times AC = AD \times CE.$$

THEOREM 15.

Circumferences of circles have to one another the ratio of their radii.

The circumference of a circle is always greater than the perimeter of a polygon inscribed in it, less than that of a polygon circumscribed about it; and each polygon may be brought perpetually nearer to the circle by increasing the number of its sides, and ultimately the two polygons and the circle will coincide.

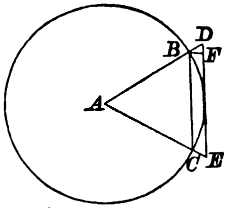




Thus in the left-hand figure the chord AB is less than AC, CB; and these are less than AD, DC, CE, FB; and these again, though more nearly coinciding with the arc AB, are less than that arc. Again, AC, CB in the right-

hand figure are greater than AD, DF, FB, and these again greater than AG, GK, KL, LN, NB, and these last, though more nearly coinciding with the arc AB, are greater than that arc.

Further, if we have two similar and regular polygons described inside and outside a circle, the difference between their perimeters may be as small as we please.



For if BC, DE be homologous sides of two such polygons, the difference between them will be DF, if BF be drawn parallel to AC. And we shall have

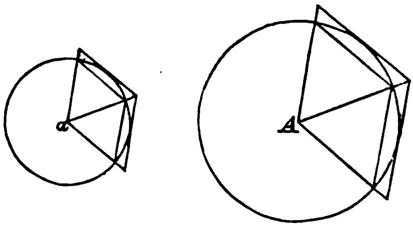
$$DF : DE :: DB : DA.$$

And this being true of all the sides is true of their sum, and therefore difference of perimeters : outer perimeter :: $DB : DA$.

Now $DB : DA$ may be made as small as we please, therefore the ratio of the difference of the perimeters to the outer perimeter may be made as small as we please; that is, since the outer perimeter does not increase but diminish at the same time, we may make the difference of perimeters as small as we please.

Now let c, C be the circumferences of two circles; r, R their radii; i, I the perimeters of two regular inscribed polygons, similar to each other; o, O the perimeters of two

regular circumscribed polygons, similar to each other and to the inscribed;



Then since the triangles in all the polygons are similar we shall have

$$r : R :: i : I,$$

and $$r : R :: o : O.$$

Now let $$r : R :: c : C'.$$

Then $$i : I :: c : C',$$

and since c is greater than i, C' is greater than I, so $o : O :: c : C'$, and since C is less than O therefore C' is less than O.

Therefore C' lies between I and O.

Now if C' differed from C by ever so small a quantity, we might have the difference between I and O still less, and then C would no longer lie between I and O, which is impossible.

So C' does not differ from C, and we have

$$r : R :: c : C.$$

Cor. Let a, A be the areas of the circles;

$$\therefore a : A :: rc : RC, \text{ p. } 36.$$

$$:: r^2 : R^2,$$

that is, circles have to each other the ratio of the squares of their radii.

SECTION III.

PROBLEMS.

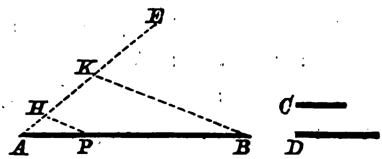
PROBLEM I.

To divide a given straight line into two parts which shall be in a given ratio.

Note. By a *given* ratio is meant the ratio of two given lines, or of two given numbers: and since two lines can always be found which have the ratio of two given numbers, it follows that a given ratio can always be represented by the ratio of two given lines.

Let AB be the given line, C and D the lines which have the given ratio; then it is required to divide AB into two parts, which have to one another the ratio of $C : D$.

Construction. From A draw a line AE making any angle with AB, and cut off parts AH, HK equal to C and



D respectively. Join KB, and draw HP parallel to KB. P will be the point of division required.

Proof. For since HP is parallel to KB,

therefore $AP : PB :: AH : HK$;

but $AH = C$; and $HK = D$;

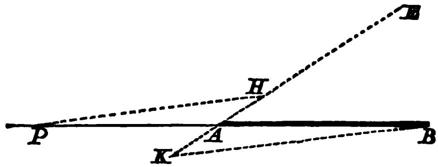
therefore $AP : PB :: C : D$,

that is, AB is divided into two parts which are to one another in the given ratio.

COR. 1. *In the same manner a line may be divided into any number of parts which have to one another given ratios.*

COR. 2. *Hence a line may be divided into any number of equal parts, the given ratios being all ratios of equality.*

Note. This construction divides the line *internally* into parts which have the given ratio. If it is required to divide



it *externally*, HK must be measured in the opposite direction along AE, as in the figure.

The proof will be the same as before.

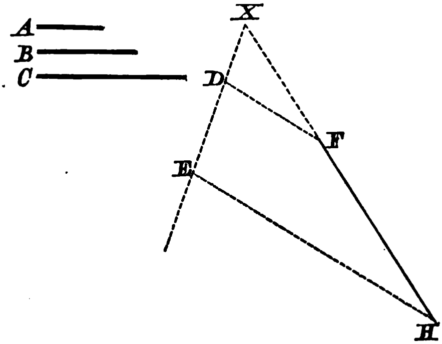
PROBLEM 2.

To find a fourth proportional to three given straight lines.

Let A, B, C be the given straight lines to which it is required to find a fourth proportional.

Construction. Take any angle X, and on one of its arms take XD, DE equal to A, B respectively: and on the other

arm take XF equal to C. Join DF, and draw EH parallel to DF, to meet XF produced in H.



Then shall FH be the line required.

Proof. For since DF is parallel to EH,

$$XD : DE :: XF : FH,$$

but XD, DE, and XF are equal to A, B, C respectively; therefore $$A : B :: C : FH;$$
that is, FH is the fourth proportional required.

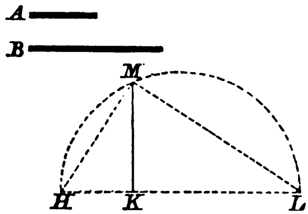
COR. *Hence a third proportional to two given straight lines can be found, by taking* $C = B$.

PROBLEM 3.

To find a mean proportional between two given straight lines.

Let A, B be the given straight lines: it is required to find a mean proportional between A and B.

Construction. Take HK, KL in the same straight line, equal to A and B respectively. On HK describe a semi-



circle, and draw KM perpendicular to HL to meet the circumference in N. KM is the line required.

Proof. Join HM, ML. Then since HML is a semicircle, HML is a right angle; therefore MK, the perpendicular from the right angle on the hypothenuse, is a mean proportional between the segments of the base; that is, MK is a mean proportional between HK and KL, or between A and B.

PROBLEM 4.

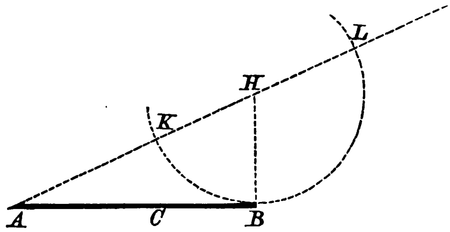
To divide a straight line in extreme and mean ratio.

A line AB is said to be divided in extreme and mean ratio in C, when

$$AB : AC :: AC : CB.$$

Let AB be the given straight line, which it is required to divide in extreme and mean ratio.

Construction. From B draw BH perpendicular to AB, and equal to half AB. Join AH. With centre H, and



radius HB describe a circle cutting AH and AH produced in K and L; and cut off from AB a part $AC = AK$.

C shall be the point required.

Proof. Since HB is at right angles to AB, AB is a tangent to the circle KBL.

Therefore $\quad AL : AB :: AB : AK.$

Therefore

$$AB : AL - AB :: AK : AB - AK \text{ (Th. 3, Cor. 3).}$$

But since $AB = 2HB = KL$, and $AK = AC$,

we have $\quad AL - AB = AK = AC,$

and $\quad AB - AK = BC;$

therefore $\quad AB : AC :: AC : BC.$

ALGEBRAICAL SOLUTION.

Remark. It is instructive to compare this with the algebraical solution. Let the given line $AB = a$, that is, contain a units of length: and let AC the portion required, which is at present unknown, $= x$.

Then by the conditions $a : x :: x : a - x$,
that is, $$a(a-x) = x^2.$$

But this is a quadratic equation in x, solving which we obtain

$$x = \frac{\pm\sqrt{5} - 1}{2} \cdot a.$$

And the geometrical construction is the expression of the operations here represented. For if $AB = a$, $HB = \frac{1}{2}a$, and $AH^2 = AB^2 + HB^2 = a^2 + \frac{1}{4}a^2 = \frac{5}{4}a^2$; and $AH = \frac{\sqrt{5}}{2} \cdot a$; and $HK = \frac{1}{2}a$; and therefore $AK = \frac{\sqrt{5}}{2}a - \frac{1}{2}a = \frac{\sqrt{5} - 1}{2}a$.

Therefore $AC = x = \frac{\sqrt{5} - 1}{2}a$.

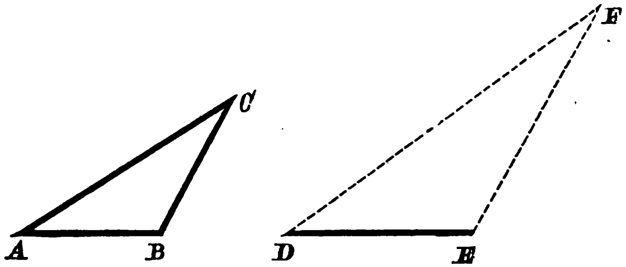
The negative sign before the radical corresponds to the solution of a problem more general than the geometrical problem as stated above, and the consideration of it must be deferred.

PROBLEM 5.

To construct a triangle similar to a given triangle, on a straight line which is to be homologous to a given side of the triangle.

Let ABC be the given triangle, DE the given straight line which is to be homologous to AB.

Construction. At D and E draw lines making with DE angles equal to the angles A and B, and let these lines meet in F. Then DEF is the triangle required.



Proof. For the triangle DEF is by construction equiangular to the triangle ABC, and therefore it is similar to it.

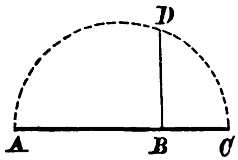
COR. *Hence a polygon can be constructed similar to any given polygon, on a straight line which is to be homologous to a given side of the polygon.*

For the given polygon can be divided into triangles.

PROBLEM 6.

To make a square which shall be to a given square in a given ratio.

Let AB be the side of the given square, $AB : BC$ in the given ratio.



Construction. On AC describe a semicircle, and draw BD perpendicular to AC, to meet the semicircle in D. BD is the side of the square required.

Proof. Since $AB : BD :: BD : BC$,
therefore $AB^2 : BD^2 :: AB : BC$,
that is $AB^2 : BD^2$ in the given ratio.

COR. *The same construction and proof are applicable to any polygon.*

For similar polygons are to one another as the squares on their homologous sides.

PROBLEM 7.

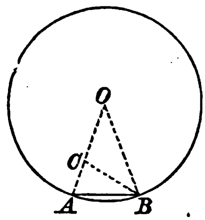
To inscribe a regular decagon in a given circle.

We shall solve this by the method of analysis and synthesis.

Analysis. The radii to two consecutive angular points must include an angle equal to $\frac{1}{10}$th of 4 right angles, or to $\frac{1}{5}$th of 2 right angles. That is, if OA, OB are such radii, the angle at O must be $\frac{1}{5}$th of two right angles, and therefore the angles at A and B must each be $\frac{2}{5}$ths of 2 right angles, since the three angles are together equal to 2 right angles.

Bisect the angle OBA by the line BC. Then OBC and CBA each are equal to AOB; and therefore $CB = CO$;

and BCA is double of COB, and therefore $= BAC$; therefore $AB = BC = CO$.



But since OBA is bisected by BC

$$OB : AB :: OC : CA;$$

therefore $$OA : OC :: OC : CA,$$

or OA is divided in extreme and mean ratio in C.

Hence the construction follows.

Synthesis. Take OA any radius of the circle; divide it in extreme and mean ratio in C so that

$$OA : OC :: OC : CA.$$

Place AB as a chord of the circle equal to OC. Join BO, CA.

Then AB is a side of a decagon inscribed in the circle.

Proof. For since $OA : OC :: OC : CA$,

and $$OA = OB, \text{ and } OC = AB;$$

therefore $$OB : BA :: OC : CA;$$

and therefore CB bisects the angle OBA; and the angle OBA is double of ABC.

And again since the triangles OAB, BAC have the angle at A common, and have the sides about the common angle proportionals, viz. $OA : AB :: AB : AC$;

Therefore these triangles are similar and equiangular; therefore the angle ABC is equal to the angle AOB.

But OBA is double of ABC, and therefore each of the angles OBA, OAB is double of AOB.

But the three angles OBA, OAB, AOB together equal 2 right angles; therefore $OAB = \frac{1}{5}$th of 2 right angles, or $\frac{1}{10}$th of 4 right angles.

Hence ten chords equal to AB could be placed round the circumference of the circle, and thus a regular decagon would be inscribed.

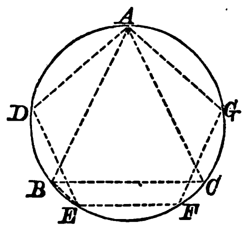
COR. 1. *Hence a regular pentagon may be inscribed in a circle by joining the alternate angular points of an inscribed decagon.*

COR. 2. *Hence regular polygons of* 20, 40, 80 *sides can be constructed.*

PROBLEM 8.

To inscribe a regular quindecagon in a given circle.

Let ABC be the given circle, in which it is required to inscribe a regular quindecagon.



Construction. Inscribe an equilateral triangle ABC, and a regular pentagon $ADEFG$, having one angular point A common.

Join BE, and place chords equal to BE round the circumference of the circle: they will form an inscribed quindecagon.

Proof. For AE is $\frac{2}{5}$ ths of the circumference, and AB is $\frac{1}{3}$ rd of the circumference; therefore BE is $\frac{2}{5} - \frac{1}{3} = \frac{1}{15}$ th of the circumference, and therefore 15 chords equal to BE can be placed round the circumference of the circle, and will form a regular quindecagon.

COR. *Hence regular polygons of* 30, 60, 120 &c. *sides can be constructed.*

Regular polygons can therefore be constructed when the number of their sides is 3, 4, 5, or 15, or these numbers multiplied by any power of 2. And besides these no other regular polygons can be constructed by the use of the ruler and compasses only, with the remarkable exception discovered by Gauss; who shewed that a polygon of $2^n + 1$ sides can be described by the ruler and compasses alone, when n is such that $2^n + 1$ is a prime number. If n has the values 1, 2, 3 ... in succession, $2^n + 1$ takes the values 3, 5, 9, 17, 33, 65, 129, 257 ... of which 3, 5, 17, 257 are primes. Hence Gauss has shewn that regular polygons of 17 and 257 sides *can* be constructed by the use of the ruler and compasses; but the construction and proof, even for the first of these, are far too tedious to be given in an elementary work.

Miscellaneous Theorems and Problems.

1. The bisector of an angle of an equilateral triangle, passes through one of the points of trisection of the perpendicular from either of the other angles on the opposite side.

2. The bisectors of the angles of a triangle intersect in one point.

3. ABC, PQR are two parallel lines such that

$$AB : BC :: PQ : QR,$$

prove that AP, BQ, CR are either parallel or meet in one point.

4. The external bisector of the vertical angle of an isosceles triangle is parallel to the base.

5. The line joining the middle points of the sides of a triangle is parallel to the base, and is equal to half the base.

6. The triangle formed by joining the middle points of the sides of a triangle is similar to the original triangle, and has one fourth of its area.

7. The lines that join the middle points of adjacent sides of a quadrilateral form a parallelogram. Under what circumstances will it be a rhombus, a square, or a rectangle?

8. CAB is a triangle, and in AC a point A' is taken, and BB' is cut off from CB produced, so that $AA' = BB'$. Prove that $A'B'$ is cut by AB into parts which have to one another the ratio $CB : CA$.

9. To inscribe a square in a triangle.

10. If two triangles are on equal bases between the same parallels any straight line parallel to their bases will cut off equivalent areas from the two triangles.

11. Make an equilateral triangle equivalent to a given square.

12. Find a point O within the triangle ABC, such that OAB, OAC, OBC shall be equivalent triangles.

13. The angle A of a triangle ABC is bisected by a line that meets the base in D: BC is bisected in O. Prove that $OB : OD :: AB + AC : AB \sim AC$.

14. Given the base, vertical angle, and ratio of the sides, construct the triangle.

15. Perpendiculars are drawn from any point within an equilateral triangle on the three sides; shew that their sum is invariable.

16. Deduce from Ptolemy's Theorem that if P is any point in the circumference of the circle circumscribing an equilateral triangle ABC, of the three lines PA, PB, PC one is equal to the sum of the other two.

17. From any point in the base of a triangle lines are drawn parallel to the two sides. Find the locus of the intersection of the diagonals of the parallelograms so formed.

18. Let P, Q be points in AB, and AB produced, so that $AP : PB :: AQ : QB$; through B draw a perpendicular to AB to meet the semicircle on PQ in M: prove that AM touches the circle at M.

19. AB is a given line, and CD a given length on a line parallel to AB, and AC, BD intersect in O; prove that as CD varies in position, the locus of O is a line parallel to AB.

20. AB is a diameter of a circle of which AEF, BEG are chords. CED is drawn through E at right angles to AB: prove that $CFDG$ is a quadrilateral such that the ratio of any pair of its adjacent sides is equal to the ratio of the other pair.

21. Divide a given arc of a circle into two parts which have their chords in a given ratio to one another.

22. If in two similar triangles lines are drawn from two of the equal angles to make equal angles with the homologous sides, these lines shall have to one another the same ratio as the sides of the triangle.

23. To make a rectilineal figure similar to a given rectilineal figure, and having a given ratio to it.

24. To find two straight lines which shall have the same ratio as two given rectangles.

25. To describe on a given straight line a rectangle equal to a given rectangle.

26. To make an isosceles triangle, with a given vertical angle, equal to a given triangle.

27. In a quadrilateral figure which cannot be inscribed in a circle the rectangle contained by the diagonals is less than the sum of the rectangles contained by the opposite sides.

28. In any triangle ABC the rectangle $AB \times AC$ is equal to the rectangle contained by the diameter of the circle circumscribing the triangle, and the perpendicular from A on BC.

29. Hence shew that if A be the area of a triangle ABC, D the diameter of the circumscribing circle,

$$A \times D = \tfrac{1}{2} AB \times BC \times CA.$$

30. Construct a rectangle equal to a given square, and having the sum of its adjacent sides equal to a given straight line.

31. Construct a rectangle equal to a given square, and having the difference of its adjacent sides equal to a given square.

32. Describe a rectangle equal to a given square, and having its sides in a given ratio.

33. To make a figure similar to a given figure, and having a given ratio to it.

34. AB is a diameter of a circle, and at A and B tangents are drawn to the circle. If PCQ be a tangent at any point C, cutting the tangents at A, B in P, Q, prove that the radius of the circle is a mean proportional between the segments PC, QC.

35. With the same figure prove that if AQ, BP intersect in R, then CR is parallel to AP or BQ.

36. If two triangles AEF, ABC have a common angle A, prove that

the triangle AEF : triangle $ABC = AE \,.\, AF : AB \,.\, AC$.

37. Given two points in a terminated straight line, find a point in the straight line such that its distances from the extremities of the line are to one another in the same ratio as its distances from the fixed points.

38. Divide a given straight line into two parts such that their squares may have a given ratio to one another.

39. AB is divided in C; shew that the perpendiculars from A, B on any straight line through C have to one another a constant ratio.

40. From the obtuse angle of a triangle to draw a line to the base which shall be a mean proportional between the segments of the base.

41. Divide a given triangle into two parts which shall have to one another a given ratio by a line parallel to one of the sides.

42. If from any point in the circumference of a circle perpendiculars be drawn to the sides, or sides produced of an inscribed triangle, prove that the feet of these perpendiculars lie in one straight line.

43. If a line be divided into any two parts to find the locus of the point in which these parts subtend equal angles.

44. If two circles touch each other externally, and also touch a straight line, prove that the part of the line between the points of contact is a mean proportional between the diameters of the circles.

45. Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

46. *ABC* is a triangle, and *O* is the point of intersection of the perpendicular from *ABC* on the opposite sides of the triangle: the circle which passes through the middle points of *OA*, *OB*, *OC*, will pass through the feet of the perpendiculars, and through the middle points of the sides of the triangle.

47. Describe a circle to touch a given straight line and a given circle, and to pass through a given point.

48. *A* and *B* are two points on the same side of a straight line which meet *AB* produced in *C*. Of all the points in this straight line find that at which *AB* subtends the greatest angle.

49. Inscribe a square in a given pentagon.

50. *ABCD* is a quadrilateral figure circumscribing a circle, and through the centre *O*, a line *EOF* equally inclined to *AB* and *BC* is drawn to meet them in *E* and *F*: prove that *AE* : *EB* :: *CF* : *FD*.

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